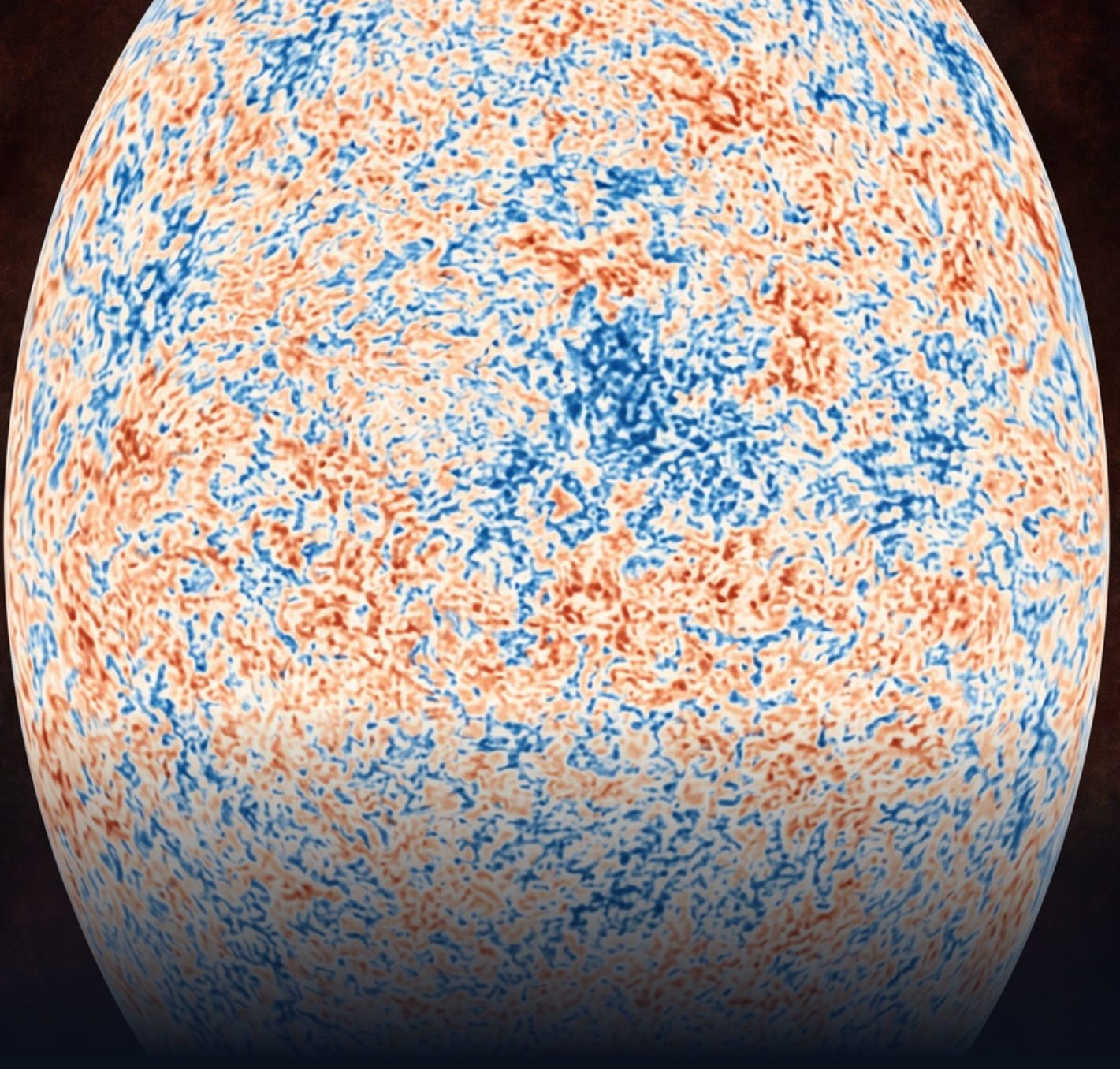




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THE EARLY UNIVERSE

# Photon-Baryon Fluid Oscillations and the Physics of the Cosmic Microwave Background Acoustic Peaks

Academic Research Paper

Theoretical Astrophysics — Step-by-Step Analytic Derivations, Vector Diagrams and Numerical Tables

# Photon-Baryon Fluid Oscillations and the Physics of the Cosmic Microwave Background Acoustic Peaks

The Early Universe

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**Abstract.** The acoustic peaks of the cosmic microwave background (CMB) are the fossil normal modes of the photon-baryon plasma before recombination. Their locations, heights, damping tail, and polarization encode the microphysics of Thomson scattering and hydrogen recombination as well as the global geometry and matter content of the Universe. This paper gives a self-contained derivation of the standard analytic theory of CMB acoustic oscillations in the tight-coupling limit. We begin with the Saha equation and show explicitly why hydrogen recombination occurs near  $T \approx 3000$  K, far below the ionization temperature  $13.6 \text{ eV}/k_B$ . We then derive the coupled photon-baryon oscillator in a Newtonian gravitational potential, identify the sound speed  $c_s = c/\sqrt{3(1+R)}$ , and solve for the effective temperature perturbation  $\Theta_0 + \Psi$ . The sound horizon fixes the harmonic wavenumbers  $k_n = n\pi/r_s$ , while baryon loading shifts the oscillator equilibrium and enhances compressional peaks. Projection to angular multipoles gives the acoustic scale and the first peak near  $\ell \approx 220$ , a classic test of spatial flatness. We also discuss Sachs-Wolfe effects, Doppler contributions, Silk damping, parameter inference, and the origin of CMB  $E$ -mode polarization.

**Keywords:** cosmic microwave background; recombination; acoustic oscillations; photon-baryon fluid; Silk damping; cosmological parameters

## 1. Introduction

The cosmic microwave background is both a photograph of the young Universe and a spectrum analyzer for the primordial plasma. Its nearly perfect blackbody spectrum records thermal equilibrium in the early hot phase, while its anisotropies at the level of  $10^{-5}$  record small departures from homogeneity. The observed angular power spectrum is dominated by a sequence of acoustic peaks, first anticipated in the theory of gravitationally driven sound waves in a radiative plasma and now measured with high precision by WMAP and Planck [7, 8, 12, 15]. The physical picture is simple but quantitatively rich: before recombination, photons and baryons were locked together by rapid Thomson scattering. Gravity compressed overdense regions, radiation pressure resisted compression, and the coupled photon-baryon medium executed acoustic oscillations. When the plasma became neutral, photons decoupled and streamed freely to us, carrying a snapshot of the oscillator phase at last scattering.

The goal of this paper is to derive this picture at a level suitable for advanced undergraduate and beginning graduate students, while retaining the mathematical precision required for research use. The emphasis is on the analytic tight-coupling model. This model is not a substitute for numerical Boltzmann solvers such as CAMB or CLASS; nevertheless, it explains why the peaks exist, why their spacing measures a sound horizon, why baryons enhance odd peaks, why small scales are exponentially damped, and why the first peak is an angular-diameter-distance test of spatial curvature. It also gives a compact conceptual map for interpreting parameter constraints such as the Planck 2018 values

$$\Omega_b h^2 \simeq 0.0224, \quad \Omega_m h^2 \simeq 0.143. \quad (1)$$

Historically, the essential ingredients came from several lines of work. The thermodynamics of ionization is encoded in the Saha equation, developed in stellar astrophysics and standard in texts such as [1]. The radiative diffusion damping scale was calculated by Silk [5]. The gravitational redshift of CMB photons was described by Sachs and Wolfe [4]. The modern analytic theory of acoustic peaks was clarified by Hu, Sugiyama, and collaborators [7, 8, 11]. Full linear perturbation theory in expanding space-time is reviewed in [13, 14, 12, 9].

We use conformal time  $\eta$ , defined by  $dt = a d\eta$ , and work mainly in Newtonian gauge with scalar perturbations. Unless otherwise stated, spatial curvature is taken to be zero when numerical estimates are quoted. Factors of  $c$  are displayed in physical interpretations and in numerical work, but many derivations use units  $c = 1$  for compactness. The symbol  $\Theta$  denotes the fractional photon temperature perturbation,

$$\Theta(\eta, \mathbf{x}, \hat{\mathbf{n}}) \equiv \frac{\Delta T}{T}, \quad (2)$$

and its monopole  $\Theta_0$  is the local isotropic temperature perturbation of the photon fluid.

## 2. Physical and Mathematical Framework

### 2.1. Metric, perturbations, and fluids

For scalar perturbations in Newtonian gauge, and neglecting anisotropic stress except where explicitly noted, we write

$$ds^2 = a^2(\eta) \left[ -(1 + 2\Psi)d\eta^2 + (1 - 2\Psi)\delta_{ij}dx^i dx^j \right]. \quad (3)$$

Here  $\Psi$  is the Newtonian gravitational potential. This one-potential convention is adequate for the tight-

coupling derivation; a fully general treatment would distinguish the time and space potentials when neutrino anisotropic stress is important.

The temperature anisotropy is expanded in Fourier modes and Legendre moments as

$$\Theta(\eta, \mathbf{k}, \mu) = \sum_{\ell=0}^{\infty} (-i)^{\ell} (2\ell+1) \Theta_{\ell}(\eta, k) P_{\ell}(\mu), \quad (4)$$

where  $\mu = \hat{\mathbf{k}} \cdot \hat{\mathbf{n}}$ . The monopole  $\Theta_0$  describes density or temperature perturbations, the dipole  $\Theta_1$  describes bulk velocity, and the quadrupole  $\Theta_2$  is the source of linear polarization.

The baryon loading parameter is defined by

$$R(\eta) \equiv \frac{3\rho_b}{4\rho_{\gamma}}. \quad (5)$$

Because  $\rho_b \propto a^{-3}$  and  $\rho_{\gamma} \propto a^{-4}$ , one has  $R \propto a$ . Thus baryon inertia becomes progressively more important as recombination is approached.

The background photon density is

$$\rho_{\gamma} = \frac{\pi^2 (k_B T)^4}{15 \hbar^3 c^5}, \quad (6)$$

and the baryon number density is related to the photon number density by

$$n_b = \eta_b n_{\gamma}, \quad n_{\gamma} = \frac{2\zeta(3)}{\pi^2} \left( \frac{k_B T}{\hbar c} \right)^3. \quad (7)$$

The small baryon-to-photon ratio  $\eta_b \simeq 6 \times 10^{-10}$  is the central reason recombination occurs at a temperature much smaller than  $13.6 \text{ eV}/k_B$ .

**Definition 1** (Last scattering and recombination). Recombination is the formation of neutral atoms from free electrons and ions. Photon decoupling, or last scattering, is the epoch at which the Thomson optical depth per Hubble time falls below order unity. The two events are close but not identical: recombination reduces the free-electron density, and decoupling occurs when the remaining scattering rate is too small to maintain local thermal contact.

## 2.2. Optical depth and visibility

Let  $\sigma_T$  be the Thomson cross section and  $n_e$  the physical free-electron density. The differential optical depth is

$$\dot{\tau} \equiv a n_e \sigma_T c, \quad (8)$$

where an overdot denotes  $d/d\eta$ . The optical depth from conformal time  $\eta$  to today  $\eta_0$  is

$$\tau(\eta) = \int_{\eta}^{\eta_0} a(\eta') n_e(\eta') \sigma_T c d\eta'. \quad (9)$$

The visibility function, the probability density that a photon last scattered at  $\eta$ , is

$$g(\eta) = \dot{\tau}(\eta) \exp[-\tau(\eta)]. \quad (10)$$

Its narrow peak defines the last-scattering surface. In the real Universe this peak occurs near

$$z_* \simeq 1089, \quad T_* \simeq 2.7255(1+z_*) \text{ K} \simeq 2970 \text{ K}. \quad (11)$$

*Remark 1.* The last-scattering surface is not infinitely thin. Its finite width smooths anisotropies on very small scales, but the dominant high- $\ell$  suppression is photon diffusion, usually called Silk damping.

## 3. Hydrogen Recombination from the Saha Equation

### 3.1. Chemical equilibrium derivation

Consider the reaction



In thermal and chemical equilibrium the chemical potentials satisfy

$$\mu_p + \mu_e = \mu_H. \quad (13)$$

For a nonrelativistic Maxwell–Boltzmann species  $i$  with mass  $m_i$ , degeneracy  $g_i$ , and number density  $n_i$ ,

$$n_i = g_i \left( \frac{m_i k_B T}{2\pi \hbar^2} \right)^{3/2} \exp \left[ \frac{\mu_i - m_i c^2}{k_B T} \right]. \quad (14)$$

Solving this expression for the exponential gives

$$\exp \left( \frac{\mu_i - m_i c^2}{k_B T} \right) = \frac{n_i}{g_i} \left( \frac{2\pi \hbar^2}{m_i k_B T} \right)^{3/2}. \quad (15)$$

Using  $m_H c^2 = m_p c^2 + m_e c^2 - \chi$ , where

$$\chi = 13.6 \text{ eV} \quad (16)$$

is the hydrogen ionization energy, equation (13) implies

$$\frac{n_p n_e}{n_H} = \frac{g_p g_e}{g_H} \left( \frac{m_e k_B T}{2\pi \hbar^2} \right)^{3/2} \exp \left( -\frac{\chi}{k_B T} \right). \quad (17)$$

We have used the approximation  $m_p m_e / m_H \simeq m_e$ . For ground-state hydrogen, the degeneracy factor is of order unity; using  $g_p g_e / g_H \simeq 1$  gives the standard pedagogical form

$$\frac{n_p n_e}{n_H} = \left( \frac{m_e k_B T}{2\pi \hbar^2} \right)^{3/2} \exp \left( -\frac{\chi}{k_B T} \right). \quad (18)$$

Let the ionization fraction be

$$x_e \equiv \frac{n_e}{n_b}. \quad (19)$$

For pure hydrogen, charge neutrality gives  $n_p = n_e = x_e n_b$ , and the neutral density is  $n_H = (1 - x_e) n_b$ . Substitution into (18) yields

$$\frac{x_e^2}{1 - x_e} = \frac{1}{n_b} \left( \frac{m_e k_B T}{2\pi \hbar^2} \right)^{3/2} \exp \left( -\frac{\chi}{k_B T} \right). \quad (20)$$

Equation (20) is the Saha equation for cosmological hydrogen recombination.

### 3.2. Why recombination occurs near 3000 K

A naive estimate might set  $k_B T = \chi$ , giving

$$T_\chi = \frac{13.6 \text{ eV}}{k_B} \simeq 1.58 \times 10^5 \text{ K}. \quad (21)$$

This is incorrect because the Universe contains enormously more photons than baryons. Even when  $k_B T \ll \chi$ , the Wien tail contains enough photons with energies above 13.6 eV to ionize hydrogen unless the Boltzmann factor is extremely small. The rough condition for recombination is not  $\exp(-\chi/k_B T) \sim 1$ , but rather

$$\eta_b^{-1} \exp\left(-\frac{\chi}{k_B T}\right) \sim 1 \quad (22)$$

up to powers of temperature. Since  $\eta_b^{-1} \sim 10^9$ , one requires

$$\frac{\chi}{k_B T} \sim \ln(10^9) \simeq 21 \quad (23)$$

before including the phase-space prefactor; the full Saha equation gives an even larger ratio,  $\chi/k_B T \simeq 40$ –50, near the epoch relevant for decoupling.

At  $T = 3000 \text{ K}$ ,

$$k_B T = 8.617 \times 10^{-5} \text{ eV K}^{-1} \times 3000 \text{ K} = 0.2585 \text{ eV}. \quad (24)$$

Hence

$$\frac{\chi}{k_B T} = \frac{13.6}{0.2585} \simeq 52.6. \quad (25)$$

The exponential is tiny,

$$\exp(-52.6) \simeq 1.4 \times 10^{-23}, \quad (26)$$

but it is multiplied by a large electron thermal phase-space density. Numerically,

$$\left(\frac{m_e k_B T}{2\pi\hbar^2}\right)^{3/2} \simeq 3.9 \times 10^{20} \text{ cm}^{-3} \quad (27)$$

at  $T = 3000 \text{ K}$ . The right-hand side of (20) is therefore of order

$$\frac{3.9 \times 10^{20} \times 1.4 \times 10^{-23}}{n_b} \simeq \frac{5.5 \times 10^{-3} \text{ cm}^{-3}}{n_b}. \quad (28)$$

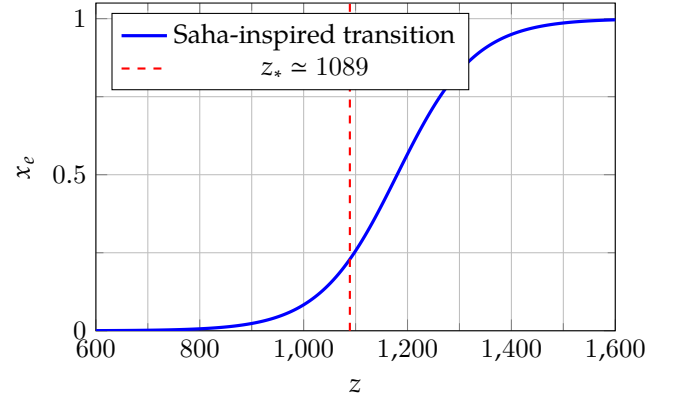
Using  $n_{\gamma,0} \simeq 410 \text{ cm}^{-3}$  and  $\eta_b \simeq 6.1 \times 10^{-10}$ ,

$$n_b(z = 1100) \simeq 6.1 \times 10^{-10} (410 \text{ cm}^{-3}) (1101)^3 \simeq 3.3 \times 10^2 \text{ cm}^{-3}. \quad (29)$$

Thus

$$\frac{x_e^2}{1 - x_e} \simeq 1.7 \times 10^{-5}, \quad x_e \simeq 4.1 \times 10^{-3} \quad (30)$$

in the simplest Saha estimate. The true recombination history is not exactly Saha equilibrium at late times because direct recombination to the ground state emits photons capable of reionizing neighboring atoms, and the bottleneck is radiative transfer through Lyman- $\alpha$  and two-photon decay channels [2, 3]. Nevertheless, the Saha calculation correctly explains the order of magnitude: recombination is delayed to thousands, not hundreds of thousands, of kelvin.



**Figure 1.** Schematic ionization fraction as a function of redshift. The curve is a smooth analytic sketch calibrated to the rapid Saha transition, not a multilevel recombination calculation. The visibility peak occurs near  $z_* \simeq 1089$ , after the free-electron abundance has fallen sharply.

## 4. Tight Coupling and the Photon–Baryon Sound Speed

### 4.1. Thomson locking

Before recombination, Thomson scattering is rapid compared with expansion and acoustic time scales:

$$\dot{\tau} \gg \mathcal{H}, \quad \dot{\tau} \gg k, \quad (31)$$

where  $\mathcal{H} = \dot{a}/a$ . In this limit photons and baryons share a common velocity,

$$v_\gamma \simeq v_b \equiv v, \quad (32)$$

and the photon quadrupole is suppressed,

$$\Theta_2 = O\left(\frac{k}{\dot{\tau}}\right) \Theta_1. \quad (33)$$

The photon pressure is

$$p_\gamma = \frac{1}{3} \rho_\gamma, \quad (34)$$

whereas baryon pressure is negligible for acoustic dynamics. For adiabatic perturbations, the total pressure perturbation is

$$\delta p = \frac{1}{3} \delta \rho_\gamma. \quad (35)$$

The photon and baryon density perturbations satisfy

$$\delta_b = \frac{3}{4} \delta_\gamma, \quad (36)$$

so the total density perturbation in the coupled fluid is

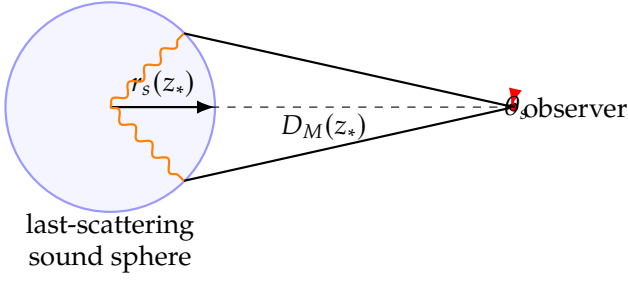
$$\delta \rho = \delta \rho_\gamma + \delta \rho_b = \delta \rho_\gamma \left(1 + \frac{3\rho_b}{4\rho_\gamma}\right) = (1 + R) \delta \rho_\gamma. \quad (37)$$

Therefore the adiabatic sound speed is

$$c_s^2 = \frac{\delta p}{\delta \rho} = \frac{1}{3(1 + R)} \quad (38)$$

in units  $c = 1$ , or

$$c_s = \frac{c}{\sqrt{3(1 + R)}}. \quad (39)$$



**Figure 2.** Geometry of the acoustic scale. The comoving sound horizon  $r_s(z_*)$  at last scattering subtends an angle  $\theta_s \approx r_s/D_M$ , where  $D_M$  is the comoving angular-diameter distance. The corresponding acoustic multipole is  $\ell_A \approx \pi D_M/r_s$ .

*Remark 2.* Baryons do not contribute much pressure, but they do contribute inertia. Increasing  $R$  lowers the sound speed and shifts the oscillator equilibrium toward compression in potential wells.

#### 4.2. Sound horizon

The comoving distance a sound wave can travel from the hot big bang to conformal time  $\eta$  is

$$r_s(\eta) = \int_0^\eta c_s(\eta') d\eta'. \quad (40)$$

Equivalently, in redshift variables,

$$r_s(z) = \int_z^\infty \frac{c_s(z')}{H(z')} dz'. \quad (41)$$

Since  $c_s$  is a physical speed and  $H^{-1}dz$  gives a comoving radial interval in units with  $c$  included in  $c_s$ , equation (41) yields a comoving length.

For a matter-plus-radiation background,

$$H^2(a) = H_0^2 [\Omega_r a^{-4} + \Omega_m a^{-3} + \Omega_\Lambda]. \quad (42)$$

At recombination, dark energy is dynamically negligible, but it matters for the angular-diameter distance to last scattering.

### 5. Derivation of the Acoustic Oscillator

#### 5.1. Continuity and Euler equations

The photon monopole is related to the photon density contrast by

$$\delta_\gamma = 4\Theta_0. \quad (43)$$

In Newtonian gauge, the photon continuity equation in Fourier space is

$$\dot{\Theta}_0 = -\frac{k}{3}v_\gamma - \dot{\Psi}. \quad (44)$$

The term  $-\dot{\Psi}$  accounts for the change in spatial curvature perturbation; it is essential for the Sachs–Wolfe and integrated Sachs–Wolfe effects.

The photon Euler equation in the tight-coupling limit may be written schematically as

$$\dot{v}_\gamma = k(\Theta_0 + \Psi) + \dot{\tau}(v_b - v_\gamma), \quad (45)$$

while the baryon Euler equation is

$$\dot{v}_b + \mathcal{H}v_b = k\Psi + \frac{\dot{\tau}}{R}(v_\gamma - v_b). \quad (46)$$

Multiplying the photon Euler equation by the photon inertia density  $4\rho_\gamma/3$ , the baryon equation by  $\rho_b$ , and adding cancels the internal drag force. With  $v_b = v_\gamma = v$ , the combined momentum equation becomes

$$\frac{d}{d\eta} [(1+R)v] = k\Theta_0 + k(1+R)\Psi. \quad (47)$$

The derivative of  $R$  incorporates the baryon Hubble drag in the expanding background.

From (44), the velocity is

$$v = -\frac{3}{k}(\dot{\Theta}_0 + \dot{\Psi}). \quad (48)$$

Substituting (48) into (47) gives

$$-\frac{3}{k} \frac{d}{d\eta} [(1+R)(\dot{\Theta}_0 + \dot{\Psi})] = k\Theta_0 + k(1+R)\Psi. \quad (49)$$

Multiplying by  $-k/3$  yields

$$\frac{d}{d\eta} [(1+R)(\dot{\Theta}_0 + \dot{\Psi})] + \frac{k^2}{3}\Theta_0 = -\frac{k^2}{3}(1+R)\Psi. \quad (50)$$

Expanding the derivative gives the desired forced oscillator:

$$\frac{d}{d\eta} [(1+R)\dot{\Theta}_0] + \frac{k^2}{3}\Theta_0 = -\frac{k^2}{3}(1+R)\Psi - \frac{d}{d\eta} [(1+R)\dot{\Psi}]. \quad (51)$$

**Proposition 1** (Tight-coupling acoustic oscillator). *In the tight-coupling limit, the photon monopole in a Newtonian potential obeys the forced oscillator equation (51). Its inertia is increased by the baryon factor  $1+R$ , its pressure term is supplied by photons, and its forcing is gravitational.*

#### 5.2. Constant-potential solution

For modes well inside matter domination, or as a first analytic approximation, take  $R$  and  $\Psi$  to be constant. Then (51) reduces to

$$(1+R)\ddot{\Theta}_0 + \frac{k^2}{3}\Theta_0 = -\frac{k^2}{3}(1+R)\Psi. \quad (52)$$

Dividing by  $1+R$  gives

$$\ddot{\Theta}_0 + k^2 c_s^2 \Theta_0 = -k^2 c_s^2 (1+R)\Psi, \quad (53)$$

where

$$c_s^2 = \frac{1}{3(1+R)}. \quad (54)$$

The equilibrium displacement is therefore

$$\Theta_{0,\text{eq}} = -(1+R)\Psi. \quad (55)$$

Define the shifted variable

$$X \equiv \Theta_0 + (1 + R)\Psi. \quad (56)$$

Since  $\Psi$  and  $R$  are constant,  $X$  satisfies

$$\ddot{X} + k^2 c_s^2 X = 0. \quad (57)$$

The solution is

$$X(\eta) = X(0) \cos(kr_s) + \frac{\dot{X}(0)}{kc_s} \sin(kr_s), \quad (58)$$

where

$$r_s = c_s \eta \quad (59)$$

for constant sound speed. For adiabatic initial conditions with negligible initial velocity,  $\dot{X}(0) = 0$ , so

$$\Theta_0 + (1 + R)\Psi = [\Theta_0(0) + (1 + R)\Psi] \cos(kr_s). \quad (60)$$

Subtracting  $R\Psi$  from both sides gives the effective observed temperature perturbation

$$\Theta_0 + \Psi = [\Theta_0(0) + (1 + R)\Psi] \cos(kr_s) - R\Psi. \quad (61)$$

This is the central analytic result. The observable monopole plus gravitational redshift oscillates around a baryon-shifted zero point  $-R\Psi$ .

### 5.3. Peaks, phases, and baryon loading

Extrema of the cosine solution occur when

$$kr_s = n\pi, \quad n = 1, 2, 3, \dots \quad (62)$$

Thus the acoustic wavenumbers are

$$k_n = \frac{n\pi}{r_s(\eta_*)}. \quad (63)$$

Odd  $n$  correspond to compression in potential wells, while even  $n$  correspond to rarefaction. Baryons shift the zero point downward in (61). If  $\Psi > 0$  labels a potential well in the present sign convention, the compressional extrema are enhanced in absolute value because the oscillator equilibrium is displaced toward compression. A schematic measure of the extrema is

$$(\Theta_0 + \Psi)_n = A(-1)^n - R\Psi, \quad (64)$$

where

$$A = \Theta_0(0) + (1 + R)\Psi. \quad (65)$$

Hence the odd–even contrast grows approximately as

$$\frac{\text{compression}}{\text{rarefaction}} \sim \frac{|A + R\Psi|}{|A - R\Psi|}. \quad (66)$$

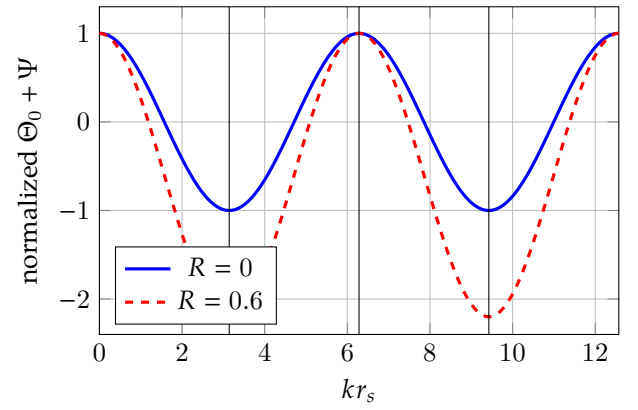
This is why the first and third peaks are particularly sensitive to the physical baryon density  $\Omega_b h^2$ .

The velocity is obtained from (48). For constant  $\Psi$ ,

$$v = -\frac{3}{k} \dot{\Theta}_0. \quad (67)$$

Using (60),

$$v = 3c_s A \sin(kr_s). \quad (68)$$



**Figure 3.** Analytic tight-coupling solution plotted against phase  $kr_s$ . The illustrative normalization uses  $\Psi = 1$  and  $A = 1 + R$ , so  $R = 0$  gives  $\cos(kr_s)$  while  $R = 0.6$  gives  $1.6 \cos(kr_s) - 0.6$ . Baryon loading shifts the zero point and enhances compressional extrema relative to rarefactions.

Thus the Doppler contribution is  $90^\circ$  out of phase with the monopole:

$$\Theta_0 + \Psi \propto \cos(kr_s), \quad v \propto \sin(kr_s). \quad (69)$$

The Doppler term contributes to temperature anisotropies through the line-of-sight projection of the velocity, but it does not by itself create the dominant peak maxima at the same phases as the monopole. Projection and angular averaging further reduce its apparent peak sharpness.

## 6. Projection to the Sky

### 6.1. Multipoles and angular-diameter distance

A Fourier mode of comoving wavenumber  $k$  at last scattering subtends an angular scale roughly

$$\theta \simeq \frac{\lambda_\perp}{D_M} \simeq \frac{1}{kD_M}, \quad (70)$$

where  $D_M$  is the comoving angular-diameter distance. Since multipole  $\ell$  corresponds to angular scale  $\theta \sim \pi/\ell$  or more roughly  $\ell \sim kD_M$ , one writes

$$\ell \simeq kD_M(z_*). \quad (71)$$

The acoustic scale is therefore

$$\ell_A = \pi \frac{D_M(z_*)}{r_s(z_*)}. \quad (72)$$

The first compressional mode has  $kr_s = \pi$ , but gravitational driving, radiation-era potential decay, neutrino stress, baryons, and projection shift the observed first temperature peak from  $\ell_A$  to approximately

$$\ell_1 \simeq 220. \quad (73)$$

For a flat universe, Planck finds  $\ell_A \simeq 301$  and the first peak near  $\ell \simeq 220$  [15]. Spatial curvature changes  $D_M$  and hence moves the entire peak pattern. The observed

location of the first peak was therefore one of the earliest precise demonstrations that the Universe is close to spatially flat.

For nonzero curvature,

$$D_M = \begin{cases} K^{-1/2} \sin(K^{1/2} \chi), & K > 0, \\ \chi, & K = 0, \\ (-K)^{-1/2} \sinh[(-K)^{1/2} \chi], & K < 0, \end{cases} \quad (74)$$

where the radial comoving distance is

$$\chi(z) = \int_0^z \frac{c \, dz'}{H(z')}. \quad (75)$$

Positive curvature decreases  $D_M$  and shifts peaks to lower  $\ell$ ; negative curvature increases  $D_M$  and shifts them to higher  $\ell$ .

## 6.2. Power spectrum representation

The temperature two-point function is customarily expressed as

$$\langle \Theta(\hat{n}) \Theta(\hat{n}') \rangle = \sum_{\ell=0}^{\infty} \frac{2\ell+1}{4\pi} C_{\ell} P_{\ell}(\hat{n} \cdot \hat{n}'). \quad (76)$$

The line-of-sight solution can be written schematically as

$$\Theta_{\ell}(k, \eta_0) = \int_0^{\eta_0} S_T(k, \eta) j_{\ell}[k(\eta_0 - \eta)] d\eta, \quad (77)$$

where  $S_T$  is the temperature source function and  $j_{\ell}$  is a spherical Bessel function. Then

$$C_{\ell} = 4\pi \int \frac{dk}{k} \mathcal{P}_{\mathcal{R}}(k) |\Theta_{\ell}(k, \eta_0)|^2. \quad (78)$$

The primordial curvature spectrum is nearly a power law,

$$\mathcal{P}_{\mathcal{R}}(k) = A_s \left( \frac{k}{k_0} \right)^{n_s-1}. \quad (79)$$

Analytic oscillator formulas explain the structure of the transfer function inside (78); precision parameter inference requires the full integral.

## 7. Large-Scale Gravitational Effects

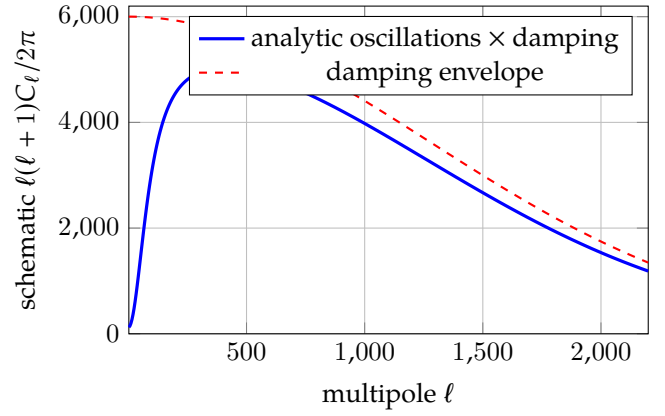
### 7.1. Ordinary Sachs–Wolfe plateau

On scales larger than the sound horizon at last scattering, pressure gradients have not had time to produce acoustic oscillations. The observed temperature perturbation is then dominated by the intrinsic photon temperature plus gravitational redshift:

$$\left( \frac{\Delta T}{T} \right)_{\text{SW}} = \Theta_0 + \Psi. \quad (80)$$

For adiabatic perturbations in matter domination, the density and potential perturbations satisfy

$$\Theta_0 = -\frac{2}{3}\Psi, \quad (81)$$



**Figure 4.** Schematic analytic temperature power spectrum, not Planck data. The curve combines an acoustic modulation, projection to  $\ell$ , a low- $\ell$  rise, and a damping envelope  $\exp[-(\ell/\ell_D)^2]$ . Real spectra require full Boltzmann evolution, gravitational driving, neutrino effects, lensing, reionization, and the finite width of last scattering.

so that

$$\Theta_0 + \Psi = \frac{1}{3}\Psi. \quad (82)$$

This is the ordinary Sachs–Wolfe effect [4]. For a nearly scale-invariant primordial spectrum, it produces the approximate low- $\ell$  plateau in

$$\frac{\ell(\ell+1)C_{\ell}}{2\pi} \simeq \text{constant} \quad (83)$$

at multipoles below the first acoustic peak.

### 7.2. Integrated Sachs–Wolfe effect

If the potential evolves while photons propagate, they gain and lose unequal gravitational energies. The integrated Sachs–Wolfe contribution is

$$\left( \frac{\Delta T}{T} \right)_{\text{ISW}} = 2 \int_{\eta_*}^{\eta_0} \dot{\Psi}[\eta, x(\eta)] d\eta \quad (84)$$

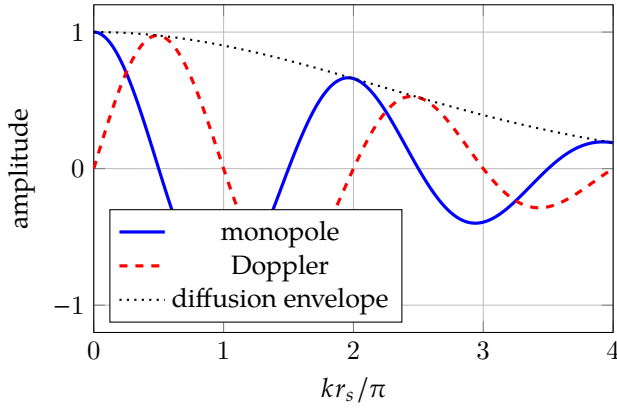
in the one-potential convention. During exact matter domination in a flat universe,  $\dot{\Psi} = 0$  and the ISW effect vanishes. It is important at early times near radiation–matter equality and at late times when dark energy causes potentials to decay.

The early ISW effect modifies the first few acoustic peaks because modes entering the horizon during radiation domination experience decaying potentials. This decay reduces gravitational confinement and drives the oscillator, enhancing acoustic amplitudes. The late ISW effect mainly affects low multipoles, where cosmic variance is large.

## 8. Diffusion Damping and the Finite Mean Free Path

### 8.1. Physical derivation

Tight coupling is an approximation. Photons actually random-walk through the plasma with physical mean



**Figure 5.** Monopole and Doppler phases in a schematic damped oscillator. The velocity contribution is shifted by  $90^\circ$  relative to the compressional temperature perturbation, and both are suppressed on scales below the diffusion length.

free path

$$\lambda_{\text{mfp}} = \frac{1}{n_e \sigma_T}. \quad (85)$$

In a time  $t$ , diffusion carries photons a physical distance

$$\lambda_D \sim \sqrt{N} \lambda_{\text{mfp}}, \quad N \sim \frac{ct}{\lambda_{\text{mfp}}}, \quad (86)$$

so

$$\lambda_D \sim \sqrt{\frac{ct}{n_e \sigma_T}}. \quad (87)$$

Perturbations on scales smaller than this are erased because photons diffuse from hot regions into cold regions. This is Silk damping [5].

A more precise tight-coupling expansion gives the comoving damping scale

$$k_D^{-2}(\eta) = \frac{1}{6} \int_0^\eta \frac{d\eta'}{\dot{\tau}} \left[ \frac{R^2}{(1+R)^2} + \frac{16}{15(1+R)} \right]. \quad (88)$$

The first term is heat conduction associated with baryon inertia, and the second is shear viscosity from the photon quadrupole. The acoustic amplitude is suppressed approximately as

$$\Theta_0(k) \rightarrow \Theta_0(k) \exp \left[ -\frac{k^2}{k_D^2} \right], \quad (89)$$

while the power is suppressed roughly by

$$C_\ell \rightarrow C_\ell \exp \left[ -2 \frac{k^2}{k_D^2} \right], \quad k \simeq \frac{\ell}{D_M}. \quad (90)$$

At recombination, the damping angular scale is of order

$$\ell_D \simeq k_D D_M \sim 10^3 - 2 \times 10^3. \quad (91)$$

This explains the rapid decline of primary temperature anisotropy at high  $\ell$ .

## 9. Polarization

Thomson scattering of anisotropic radiation produces linear polarization. The differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{3\sigma_T}{8\pi} |\hat{\epsilon}' \cdot \hat{\epsilon}|^2, \quad (92)$$

so outgoing polarization depends on the angular distribution of incident radiation. A pure monopole produces no preferred direction, and a pure dipole can be transformed away locally by a boost. The leading source is the quadrupole:

$$\Pi \simeq \Theta_2 + \Theta_{P,0} + \Theta_{P,2}. \quad (93)$$

In tight coupling,  $\Theta_2$  is small, but near recombination the photon mean free path grows and a quadrupole develops. This produces  $E$ -mode polarization with acoustic structure.

Scalar perturbations generate only gradient-type polarization at linear order:

$$B_{\ell m} = 0 \quad \text{for scalar modes at first order.} \quad (94)$$

The  $E$ -mode peaks are closely related to the velocity field and are therefore approximately out of phase with the temperature monopole. The temperature-polarization cross spectrum alternates sign according to the relative phase of density and velocity perturbations. Detection of this phase pattern is a powerful confirmation that the peaks are coherent acoustic oscillations rather than random late-time structures [10, 11].

## 10. Worked Numerical Applications

### 10.1. Baryon loading at last scattering

Using the Planck 2018 physical densities

$$\Omega_b h^2 = 0.0224, \quad \Omega_m h^2 = 0.143, \quad (95)$$

the baryon loading can be estimated by

$$R(z) \simeq 31.5 \Omega_b h^2 \left( \frac{T_0}{2.725 \text{ K}} \right)^{-4} \left( \frac{1000}{z} \right). \quad (96)$$

At  $z_* = 1089$  this gives

$$R_* \simeq 31.5(0.0224) \left( \frac{1000}{1089} \right) \simeq 0.65. \quad (97)$$

Therefore the sound speed at last scattering is

$$c_{s,*} = \frac{c}{\sqrt{3(1+0.65)}} = 0.45 c. \quad (98)$$

The baryons have reduced the sound speed from  $c/\sqrt{3} = 0.577c$  to about  $0.45c$  and shifted the oscillator equilibrium substantially.

## 10.2. Sound horizon and acoustic scale

A useful fitting value for the comoving sound horizon at photon decoupling is

$$r_s(z_*) \simeq 144.6 \text{ Mpc}. \quad (99)$$

The comoving angular-diameter distance in a flat Planck-like cosmology is approximately

$$D_M(z_*) \simeq 1.39 \times 10^4 \text{ Mpc}. \quad (100)$$

The angle subtended by the sound horizon is

$$\theta_s = \frac{r_s}{D_M} = \frac{144.6}{1.39 \times 10^4} = 1.04 \times 10^{-2} \text{ rad}. \quad (101)$$

In degrees,

$$\theta_s = 1.04 \times 10^{-2} \frac{180}{\pi} \simeq 0.60^\circ. \quad (102)$$

The acoustic multipole is

$$\ell_A = \pi \frac{D_M}{r_s} = \pi \frac{1.39 \times 10^4}{144.6} \simeq 302. \quad (103)$$

The observed first temperature peak is not exactly at  $\ell_A$  but near

$$\ell_1 \simeq 220, \quad (104)$$

because the first peak is shifted by radiation driving, projection, baryon loading, and neutrino-induced phase effects.

## 10.3. Matter–radiation equality

The redshift of matter–radiation equality is controlled primarily by  $\Omega_m h^2$ . With photons and three standard neutrino species,

$$1 + z_{\text{eq}} \simeq \frac{\Omega_m h^2}{\Omega_r h^2} \simeq 3400 \left( \frac{\Omega_m h^2}{0.143} \right). \quad (105)$$

The corresponding horizon scale is often expressed as

$$k_{\text{eq}} \simeq 0.010 \text{ Mpc}^{-1} \left( \frac{\Omega_m h^2}{0.143} \right). \quad (106)$$

Changing  $\Omega_m h^2$  changes when potentials decay. This alters early ISW driving and therefore the first-to-third peak structure. Thus  $\Omega_m h^2$  is inferred not only from the peak positions but from their relative heights.

## 10.4. Silk damping scale

A representative comoving photon diffusion scale at recombination is

$$k_D^{-1}(z_*) \simeq 4 \text{ Mpc}, \quad (107)$$

or

$$k_D(z_*) \simeq 0.25 \text{ Mpc}^{-1}. \quad (108)$$

Projected to the sky, this corresponds to

$$\ell_D \simeq k_D D_M \simeq 0.25(1.39 \times 10^4) \simeq 3.5 \times 10^3. \quad (109)$$

Because visibility width, projection, and the precise definition of the damping envelope modify this simple estimate, the observed damping tail becomes prominent already for

$$\ell \gtrsim 1000. \quad (110)$$

**Table 1.** Representative numerical scales in a Planck-like flat  $\Lambda$ CDM cosmology.

| Quantity       | Approximate value              | Physical role          |
|----------------|--------------------------------|------------------------|
| $z_*$          | 1089                           | photon decoupling      |
| $T_*$          | $2.97 \times 10^3 \text{ K}$   | recombination era      |
| $\Omega_b h^2$ | 0.0224                         | baryon loading         |
| $\Omega_m h^2$ | 0.143                          | equality, potentials   |
| $R_*$          | 0.65                           | inertia ratio          |
| $r_s(z_*)$     | 144.6 Mpc                      | standard ruler         |
| $D_M(z_*)$     | $1.39 \times 10^4 \text{ Mpc}$ | projection distance    |
| $\ell_A$       | 301–302                        | acoustic scale         |
| $\ell_1$       | $\simeq 220$                   | first temperature peak |

## 11. How Peak Heights Determine Cosmological Parameters

The peak locations are primarily a ruler-and-distance measurement, but the peak heights measure physical densities. The baryon density controls  $R$ :

$$R_* \propto \Omega_b h^2. \quad (111)$$

Larger  $R_*$  increases baryon inertia and shifts the oscillator equilibrium as in (61). It enhances compressional peaks relative to rarefaction peaks:

$$\frac{C_{\ell_1}}{C_{\ell_2}} \text{ increases with } \Omega_b h^2. \quad (112)$$

This is the clean physical reason that the first-to-second peak ratio is a baryometer.

The matter density controls equality:

$$1 + z_{\text{eq}} \propto \Omega_m h^2. \quad (113)$$

If equality occurs later, more acoustic modes enter the horizon during radiation domination, when potentials decay. Potential decay drives oscillations and changes the early ISW contribution. Therefore the relative heights of the first several peaks constrain  $\Omega_m h^2$ .

The primordial amplitude and tilt enter through

$$C_\ell \propto A_s \left( \frac{k}{k_0} \right)^{n_s-1} |T_\ell(k)|^2, \quad (114)$$

where  $T_\ell$  is the radiation transfer function. Reionization suppresses primary temperature anisotropy at small angular scales by approximately

$$C_\ell \rightarrow e^{-2\tau_{\text{re}}} C_\ell \quad (115)$$

for modes well inside the horizon at reionization, creating a degeneracy between  $A_s$  and the optical depth  $\tau_{\text{re}}$  that is broken by large-angle polarization.

## 12. Discussion

The derivation above rests on several controlled approximations. First, tight coupling assumes  $\dot{\tau} \gg k, \mathcal{H}$

and therefore suppresses photon quadrupoles, polarization, and viscosity. This is excellent at early times but fails near recombination, precisely when polarization and diffusion damping are generated. The damping scale (88) is obtained by expanding beyond leading order in  $k/\tau$ .

Second, the constant-potential solution (61) captures the phase and baryon loading but omits gravitational driving by time-dependent potentials. In radiation domination, photon pressure prevents clustering of the dominant radiation component on subhorizon scales, and potentials decay. This decay acts as a driving term in (51) and boosts acoustic amplitudes. Free-streaming neutrinos also carry anisotropic stress and shift the acoustic phase. These effects are essential for precision cosmology but do not obscure the basic oscillator interpretation.

Third, recombination is not exactly given by the Saha equation. The Saha treatment assumes instantaneous chemical equilibrium. Real cosmological recombination is delayed by the Lyman- $\alpha$  escape probability and by the metastable  $2s \rightarrow 1s$  two-photon decay. Peebles and Zeldovich-type effective three-level atom calculations improve the result, and modern multilevel recombination codes are required for Planck-level accuracy [2, 3, 15]. Nevertheless, the Saha equation explains why the relevant temperature is  $T \simeq 3000$  K rather than  $1.6 \times 10^5$  K.

Fourth, the mapping  $\ell \simeq kD_M$  is approximate. The exact mapping is performed by spherical Bessel functions in (77). The finite thickness of last scattering, gravitational lensing, Doppler terms, polarization, reionization, and late-time ISW effects all modify the observed spectra. Gravitational lensing smooths acoustic peaks and converts some  $E$  polarization into  $B$  polarization. Foregrounds and secondary anisotropies, including the thermal and kinetic Sunyaev–Zeldovich effects, dominate some small-scale observations and must be modeled in real analyses.

The comparison with observations is extraordinarily successful. The measured peak phases confirm coherent adiabatic initial conditions, as expected from simple inflationary models. The first peak position indicates near spatial flatness. The odd–even peak contrast gives  $\Omega_b h^2 \simeq 0.0224$ , in agreement with primordial nucleosynthesis. The overall peak pattern yields  $\Omega_m h^2 \simeq 0.143$ , fixing the redshift of equality. The damping tail constrains the helium fraction, the spectral index, the expansion rate, and any extra relativistic energy density. Polarization confirms the velocity phase and measures the optical depth to reionization.

### 13. Conclusion

The acoustic peaks of the CMB arise from a simple physical system: a gravitationally forced photon–baryon oscillator observed at a special time, last scattering. The Saha equation explains why the plasma becomes neutral near  $z \simeq 1100$ , because the enormous photon-to-

baryon ratio allows the high-energy tail of the black-body distribution to maintain ionization until  $k_B T \ll 13.6$  eV. Before this epoch, Thomson scattering enforces tight coupling, and the coupled fluid has sound speed

$$c_s = \frac{c}{\sqrt{3(1+R)}}. \quad (116)$$

Combining photon continuity with the total photon–baryon Euler equation gives the oscillator

$$\frac{d}{d\eta} [(1+R)\dot{\Theta}_0] + \frac{k^2}{3}\Theta_0 = -\frac{k^2}{3}(1+R)\Psi - \frac{d}{d\eta} [(1+R)\dot{\Psi}], \quad (117)$$

whose constant-potential solution is

$$\Theta_0 + \Psi = [\Theta_0(0) + (1+R)\Psi] \cos(kr_s) - R\Psi. \quad (118)$$

The phase condition  $k_n r_s = n\pi$  gives the harmonic structure, while projection gives  $\ell \simeq kD_M$ . Baryons enhance odd peaks, matter density controls potential decay and equality, diffusion damping erases small scales, and polarization reveals the quadrupole generated at decoupling. These linked phenomena make the CMB an exceptionally precise probe of both microphysics and cosmic geometry.

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