



ACCRETION DISKS

The Shakura-Sunyaev Thin Disk Model and the Flux Equations of Viscous Energy Dissipation

Academic Research Paper

Theoretical Astrophysics — Step-by-Step Analytic Derivations, Vector Diagrams and Numerical Tables

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Abstract. The Shakura–Sunyaev geometrically thin, optically thick accretion disk is the basic analytic model connecting mass accretion, angular-momentum transport, viscous dissipation, and thermal radiation in compact astrophysical systems. This paper develops the model from first principles, beginning with the gravitational energy scale, Eddington limit, Keplerian shear, and vertically integrated conservation laws. We derive the steady-state surface-density relation with a zero-torque inner boundary, the viscous dissipation law, the effective-temperature distribution, the bolometric luminosity, and the multi-colour blackbody spectrum. The derivations emphasize the physical meaning of each term: local dissipation is set by shear, global luminosity by binding energy, and angular-momentum transport by an outward viscous torque. We then introduce the α prescription, hydrostatic scale height, diffusion equation for the surface density, and standard radial scalings of the Shakura–Sunyaev solution. Worked examples for cataclysmic variables, X-ray binaries, and active galactic nuclei illustrate the numerical magnitude of temperatures, luminosities, time-scales, and efficiencies. We conclude by discussing observational diagnostics, the Soltan efficiency argument, magnetorotational turbulence as the physical origin of effective viscosity, and limitations associated with radiation-pressure instability, relativistic inner-boundary physics, and disk winds.

Keywords: accretion disks; Shakura–Sunyaev model; viscous dissipation; blackbody spectra; Eddington luminosity; angular momentum transport

1. Introduction

Accretion is among the most efficient known mechanisms for converting gravitational binding energy into radiation. In close binaries, young stellar objects, active galactic nuclei, and transient X-ray sources, matter with nonzero angular momentum cannot fall radially onto the central object. Instead it forms a differentially rotating disk in which angular momentum is transported outward while mass drifts inward. The resulting release of orbital binding energy powers optical, ultraviolet, and X-ray luminosities over a range extending from dwarf novae to quasars.

The classical thin-disk model of Shakura and Sunyaev [15], developed in parallel with the global viscous formulation of Lynden-Bell and Pringle [10], remains the simplest self-consistent theory of an optically thick accretion flow. Its assumptions are severe but powerful: the disk is geometrically thin, $H/r \ll 1$; the angular velocity is nearly Keplerian; the mass accretion rate is steady; internal stresses can be represented by an effective viscosity; and heat generated by differential rotation is radiated locally from the two disk faces. Under these assumptions the model predicts the radial flux law

$$D(r) = \frac{3GM\dot{M}}{8\pi r^3} \left[1 - \left(\frac{R_{\text{in}}}{r} \right)^{1/2} \right], \quad (1)$$

where D is the radiative flux from one surface, M is the central mass, $\dot{M}$ is the inward mass accretion rate, and R_{in} is the inner edge. Equation (1) is one of the most widely used formulae in high-energy astrophysics. It underlies estimates of accretion rates in cataclysmic variables, thermal-state black-hole binaries,

and quasars; it is also the Newtonian limit of relativistic disk theory [11, 12].

The purpose of this paper is pedagogical and theoretical: to give a complete derivation of the flux equations of viscous energy dissipation, with all intermediate steps explicit. We follow the notation customary in graduate texts [6, 7, 13, 3]. The derivation begins with conservation of mass and angular momentum, not with the final flux law. This route is essential because the disk luminosity is not obtained merely by differentiating the gravitational potential energy. Half of the released gravitational energy remains as orbital kinetic energy until it is disposed of at the stellar surface, boundary layer, innermost stable circular orbit, or event horizon. The other half is dissipated in the extended disk, but with a radial distribution modified by the inner-boundary torque.

Thin disks also introduce a useful hierarchy of time-scales. The dynamical time is orbital; the thermal time is longer by a factor α^{-1} ; and the viscous time is longer still by $(r/H)^2$. This separation justifies hydrostatic vertical equilibrium and local thermal balance, while allowing secular evolution of the surface density. Yet the model also hides deep physics in the single parameter α . Modern magnetohydrodynamic simulations identify the magnetorotational instability as the likely source of turbulent stress [1, 2], but the saturation of that turbulence and its relation to radiation transport remain active research problems.

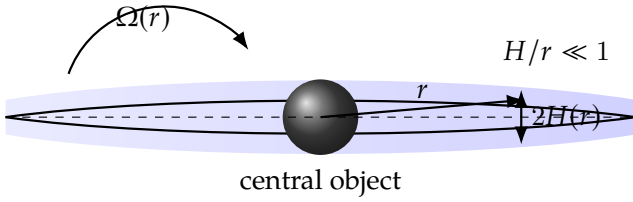


Figure 1. Meridional cross-section of a geometrically thin disk. Vertical hydrostatic balance gives the scale height $H = c_s/\Omega_K$, while radial force balance is nearly Keplerian because $H/r \ll 1$.

2. Physical and Mathematical Framework

2.1. Accretion luminosity and efficiency

A mass element m falling from infinity to radius R in the Newtonian potential of a mass M changes its gravitational potential energy by

$$\Delta E_{\text{grav}} = \frac{GMm}{R}. \quad (2)$$

If matter arrives at a rate $\dot{M}$, the natural accretion power scale is therefore

$$L_{\text{acc}} = \frac{GMM\dot{M}}{R}. \quad (3)$$

This estimate is deliberately general: it is the maximum Newtonian power available if the matter comes to rest at R . It applies approximately to accretion onto a material stellar surface.

It is useful to express the luminosity as a fraction of rest-mass energy:

$$L = \eta \dot{M} c^2, \quad (4)$$

where η is the radiative efficiency. Combining (3) and (4) gives the Newtonian surface efficiency

$$\eta_{\text{surf}} = \frac{GM}{Rc^2} = \frac{1}{2} \frac{R_S}{R}, \quad (5)$$

with Schwarzschild radius

$$R_S = \frac{2GM}{c^2}. \quad (6)$$

For a white dwarf $\eta \sim 10^{-4}$, for a neutron star $\eta \sim 0.1$, and for a black hole the relativistic efficiency is set by the binding energy at the innermost stable circular orbit.

A circular Keplerian orbit has total specific mechanical energy

$$e(r) = \frac{v_\phi^2}{2} - \frac{GM}{r}. \quad (7)$$

Using $v_\phi^2 = GM/r$ gives

$$e(r) = -\frac{GM}{2r}. \quad (8)$$

Thus a thin disk extending to R_{in} can radiate only

$$L_{\text{disk}} = \frac{GMM\dot{M}}{2R_{\text{in}}} \quad (9)$$

if the stress vanishes at the inner edge. The missing half of (3) is not missing physically: it is orbital kinetic energy at R_{in} .

2.2. Eddington luminosity

The Eddington limit follows from the balance between outward radiation force and inward gravity for ionized gas. At radius r , the radiative flux corresponding to luminosity L is

$$F = \frac{L}{4\pi r^2}. \quad (10)$$

For electron scattering opacity κ , the outward force per unit mass is

$$f_{\text{rad}} = \frac{\kappa F}{c} = \frac{\kappa L}{4\pi r^2 c}. \quad (11)$$

The inward gravitational force per unit mass is

$$f_{\text{grav}} = \frac{GM}{r^2}. \quad (12)$$

Setting (11) equal to (12) gives

$$L_{\text{Edd}} = \frac{4\pi GMc}{\kappa}. \quad (13)$$

For fully ionized hydrogen $\kappa = \sigma_T/m_p$, hence

$$L_{\text{Edd}} = \frac{4\pi GMm_p c}{\sigma_T}. \quad (14)$$

Numerically,

$$L_{\text{Edd}} = 1.26 \times 10^{38} \left(\frac{M}{M_\odot} \right) \text{erg s}^{-1}. \quad (15)$$

The corresponding Eddington accretion rate is defined by

$$\dot{M}_{\text{Edd}} = \frac{L_{\text{Edd}}}{\eta c^2}. \quad (16)$$

For $\eta = 0.1$,

$$\dot{M}_{\text{Edd}} = 2.2 \times 10^{-8} \left(\frac{M}{M_\odot} \right) M_\odot \text{yr}^{-1}. \quad (17)$$

2.3. Keplerian angular velocity and shear

Radial force balance for a cold circular orbit is

$$\frac{v_\phi^2}{r} = \frac{GM}{r^2}. \quad (18)$$

With $v_\phi = r\Omega$, the Keplerian angular velocity is

$$\Omega_K(r) = \left(\frac{GM}{r^3} \right)^{1/2}. \quad (19)$$

The specific angular momentum is

$$j(r) = r^2 \Omega_K = (GMr)^{1/2}. \quad (20)$$

Differentiating (19) gives the shear

$$\frac{d\Omega_K}{dr} = -\frac{3}{2} \frac{\Omega_K}{r}. \quad (21)$$

The negative sign is essential: inner rings rotate faster than outer rings. Viscous or turbulent stress therefore transports angular momentum outward and mechanical energy both outward and into heat.

Definition 1. A thin disk is a rotating gaseous configuration satisfying

$$\frac{H}{r} \ll 1, \quad |v_r| \ll v_\phi, \quad \Omega \simeq \Omega_K. \quad (22)$$

It is optically thick if its vertical optical depth obeys

$$\tau = \frac{1}{2} \kappa \Sigma \gg 1, \quad (23)$$

where Σ is surface mass density.

3. Conservation Laws for a Viscous Thin Disk

3.1. Surface density and mass conservation

The surface density is the vertical integral

$$\Sigma(r, t) = \int_{-\infty}^{\infty} \rho(r, z, t) dz. \quad (24)$$

Axisymmetric mass conservation in cylindrical coordinates is

$$\frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (r \rho v_r) + \frac{\partial}{\partial z} (\rho v_z) = 0. \quad (25)$$

Integrating (25) over z , and assuming no net mass flux through the upper and lower surfaces, gives

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (r \Sigma v_r) = 0. \quad (26)$$

The mass accretion rate is conventionally positive inward:

$$\dot{M}(r, t) = -2\pi r \Sigma v_r. \quad (27)$$

Because accretion has $v_r < 0$, equation (27) gives $\dot{M} > 0$. In steady state, $\partial \Sigma / \partial t = 0$ and (26) implies

$$\frac{d}{dr} (r \Sigma v_r) = 0, \quad (28)$$

so that

$$\dot{M} = \text{constant}. \quad (29)$$

3.2. Viscous torque

Let ν denote the effective kinematic viscosity. The relevant shear stress is

$$T_{r\phi} = \rho \nu r \frac{d\Omega}{dr}. \quad (30)$$

Vertical integration gives the stress resultant

$$W_{r\phi} = \nu \Sigma r \frac{d\Omega}{dr}. \quad (31)$$

The torque exerted across a cylindrical surface of radius r is force times lever arm. The area of the two disk faces

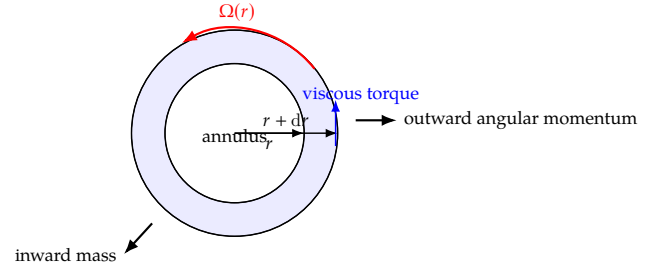


Figure 2. Angular-momentum transport in a viscous annulus. Faster inner gas exerts a forward torque on outer gas; outer gas exerts a backward torque on inner gas. The net effect is outward angular-momentum transport and inward mass drift.

projected through the cylindrical cut is $2\pi r$ times vertical integration already contained in $W_{r\phi}$, hence

$$G(r) = 2\pi r \cdot r \cdot W_{r\phi}. \quad (32)$$

Therefore

$$G(r) = 2\pi r \nu \Sigma r^2 \frac{d\Omega}{dr}. \quad (33)$$

Since $d\Omega/dr < 0$, this expression is negative with the sign convention of the local stress. It is often clearer to use the outward angular-momentum flux magnitude

$$\mathcal{G}(r) = -2\pi r \nu \Sigma r^2 \frac{d\Omega}{dr}. \quad (34)$$

For a Keplerian disk, (21) gives

$$\mathcal{G}(r) = 3\pi \nu \Sigma r^2 \Omega_K. \quad (35)$$

3.3. Angular-momentum conservation

The angular momentum per unit area is $\Sigma r^2 \Omega$. Conservation gives

$$\frac{\partial}{\partial t} (\Sigma r^2 \Omega) + \frac{1}{r} \frac{\partial}{\partial r} (r \Sigma v_r r^2 \Omega) = \frac{1}{2\pi r} \frac{\partial \mathcal{G}}{\partial r}. \quad (36)$$

The right side is the divergence of the viscous torque. If the potential is fixed, $\Omega = \Omega(r)$ and not an independent thermodynamic variable. Multiplying (26) by $r^2 \Omega$ and subtracting it from (36) yields

$$\Sigma v_r \frac{d}{dr} (r^2 \Omega) = \frac{1}{2\pi r} \frac{d\mathcal{G}}{dr}. \quad (37)$$

Using (27), this becomes

$$-\dot{M} \frac{dj}{dr} = \frac{d\mathcal{G}}{dr}. \quad (38)$$

Integrating with respect to radius gives

$$\dot{M} j - \mathcal{G} = C, \quad (39)$$

where C is a constant angular-momentum flux.

At the inner edge R_{in} the zero-torque condition is

$$\mathcal{G}(R_{\text{in}}) = 0. \quad (40)$$

Then (39) evaluated at R_{in} gives

$$C = \dot{M} j_{\text{in}}, \quad (41)$$

where

$$j_{\text{in}} = (GMR_{\text{in}})^{1/2}. \quad (42)$$

Therefore

$$\mathcal{G} = \dot{M}(j - j_{\text{in}}). \quad (43)$$

Substituting (35) into (43) gives

$$3\pi\nu\Sigma r^2\Omega_K = \dot{M} [(GMr)^{1/2} - (GMR_{\text{in}})^{1/2}]. \quad (44)$$

Since $r^2\Omega_K = (GMr)^{1/2}$, division by $3\pi(GMr)^{1/2}$ yields the central steady-state relation

$$\nu\Sigma = \frac{\dot{M}}{3\pi} \left[1 - \left(\frac{R_{\text{in}}}{r} \right)^{1/2} \right]. \quad (45)$$

Remark 1. Equation (45) does not say that Σ vanishes at the inner edge. It says that the product $\nu\Sigma$ vanishes if the torque vanishes. In realistic disks the stress may not be exactly zero near a stellar boundary layer, magnetosphere, or plunging relativistic region.

4. Viscous Dissipation and the Flux Law

4.1. Local shear heating

The rate of viscous dissipation per unit volume in a shear flow is stress times strain rate. For differential rotation,

$$q^+ = \rho\nu \left(r \frac{d\Omega}{dr} \right)^2. \quad (46)$$

Integrating over height and dividing equally between the two radiating faces gives the flux from one disk face:

$$D(r) = \frac{1}{2} \nu \Sigma \left(r \frac{d\Omega}{dr} \right)^2. \quad (47)$$

For Keplerian rotation, (21) gives

$$r \frac{d\Omega_K}{dr} = -\frac{3}{2} \Omega_K. \quad (48)$$

Therefore

$$D(r) = \frac{1}{2} \nu \Sigma \left(\frac{9}{4} \Omega_K^2 \right), \quad (49)$$

or

$$D(r) = \frac{9}{8} \nu \Sigma \Omega_K^2. \quad (50)$$

Substituting (45) and $\Omega_K^2 = GM/r^3$ gives

$$D(r) = \frac{3GM\dot{M}}{8\pi r^3} \left[1 - \left(\frac{R_{\text{in}}}{r} \right)^{1/2} \right]. \quad (51)$$

This is the Shakura–Sunyaev flux equation. The factor in square brackets expresses the loss of local dissipation near the inner edge when the viscous torque there is zero. Far from the inner edge the local flux approaches the result expected from the radial release of orbital binding energy.

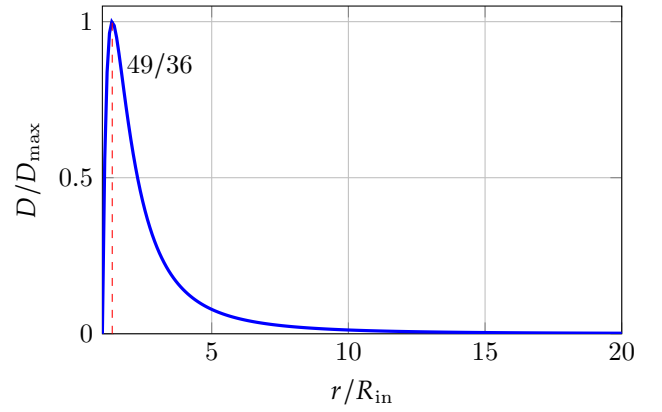


Figure 3. Normalized viscous flux $D(r)$ from equation (51). The flux vanishes at R_{in} for a zero-torque boundary and reaches its maximum at $r = (49/36)R_{\text{in}}$.

4.2. Effective temperature

If each annulus radiates locally as a blackbody from each surface, then

$$D(r) = \sigma T_{\text{eff}}^4(r), \quad (52)$$

where σ is the Stefan–Boltzmann constant. Thus

$$T_{\text{eff}}(r) = \left[\frac{3GM\dot{M}}{8\pi\sigma r^3} \left(1 - \sqrt{\frac{R_{\text{in}}}{r}} \right) \right]^{1/4}. \quad (53)$$

For $r \gg R_{\text{in}}$,

$$T_{\text{eff}}(r) \propto r^{-3/4}. \quad (54)$$

The maximum temperature occurs where $D(r)$ is maximum. Let

$$x = \frac{r}{R_{\text{in}}}, \quad f(x) = x^{-3}(1 - x^{-1/2}). \quad (55)$$

Differentiating,

$$\frac{df}{dx} = -3x^{-4}(1 - x^{-1/2}) + \frac{1}{2}x^{-9/2}. \quad (56)$$

Setting $df/dx = 0$ gives

$$-3 + \frac{7}{2}x^{-1/2} = 0. \quad (57)$$

Hence

$$x^{1/2} = \frac{7}{6}, \quad (58)$$

and

$$r_{\text{max}} = \frac{49}{36} R_{\text{in}}. \quad (59)$$

The maximum temperature is

$$T_{\text{max}} = 0.488 \left(\frac{3GM\dot{M}}{8\pi\sigma R_{\text{in}}^3} \right)^{1/4}. \quad (60)$$

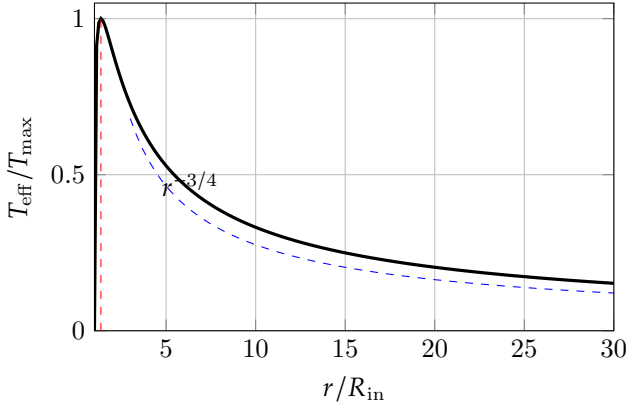


Figure 4. Normalized effective temperature from equation (53). At large radius the zero-torque correction is negligible and $T_{\text{eff}} \propto r^{-3/4}$.

4.3. Integrated luminosity and the other half of the energy

The luminosity emitted by both disk faces from an annulus of width dr is

$$dL = 2(2\pi r dr)D(r). \quad (61)$$

Thus

$$L_{\text{disk}} = 4\pi \int_{R_{\text{in}}}^{\infty} r D(r) dr. \quad (62)$$

Substituting (51),

$$L_{\text{disk}} = \frac{3GM\dot{M}}{2} \int_{R_{\text{in}}}^{\infty} r^{-2} \left[1 - \left(\frac{R_{\text{in}}}{r} \right)^{1/2} \right] dr. \quad (63)$$

The first integral is

$$\int_{R_{\text{in}}}^{\infty} r^{-2} dr = \frac{1}{R_{\text{in}}}, \quad (64)$$

and the second is

$$R_{\text{in}}^{1/2} \int_{R_{\text{in}}}^{\infty} r^{-5/2} dr = \frac{2}{3} \frac{1}{R_{\text{in}}}. \quad (65)$$

Therefore

$$L_{\text{disk}} = \frac{3GM\dot{M}}{2} \left(\frac{1}{R_{\text{in}}} - \frac{2}{3} \frac{1}{R_{\text{in}}} \right), \quad (66)$$

so that

$$L_{\text{disk}} = \frac{GM\dot{M}}{2R_{\text{in}}}. \quad (67)$$

This equals the decrease in Keplerian orbital energy between infinity and R_{in} . The remaining energy is the kinetic energy advected through the inner boundary:

$$\dot{E}_{\text{kin,in}} = \frac{1}{2} \dot{M} v_K^2(R_{\text{in}}) = \frac{GM\dot{M}}{2R_{\text{in}}}. \quad (68)$$

If the accretor has a material surface rotating slowly compared with the Keplerian speed, this energy is dissipated in a boundary layer. If the accretor is a black hole, much of it crosses the horizon after relativistic processing in the plunging region.

Proposition 1. For a Newtonian Keplerian thin disk with a zero-torque inner boundary, the total bolometric luminosity from the disk is given by equation (67), independent of the detailed viscosity law.

5. The Multi-Colour Blackbody Spectrum

5.1. Spectral integral

The specific intensity of a blackbody is

$$B_\nu(T) = \frac{2h\nu^3}{c^2} \frac{1}{\exp(h\nu/kT) - 1}. \quad (69)$$

For a disk viewed at inclination i and distance d , the observed flux density is

$$F_\nu = \frac{\cos i}{d^2} \int 2\pi r B_\nu[T_{\text{eff}}(r)] dr. \quad (70)$$

Including both sides would be inappropriate for a distant observer seeing one face; luminosity spectra, by contrast, include both faces. For scaling arguments constants are unimportant. At radii well outside R_{in} ,

$$T(r) = T_0 \left(\frac{r}{R_0} \right)^{-3/4}. \quad (71)$$

Define the radius r_ν by

$$kT(r_\nu) = h\nu. \quad (72)$$

From (71),

$$r_\nu = R_0 \left(\frac{kT_0}{h\nu} \right)^{4/3}. \quad (73)$$

Now set

$$u = \frac{h\nu}{kT(r)} = \left(\frac{r}{r_\nu} \right)^{3/4}. \quad (74)$$

Then

$$r = r_\nu u^{4/3}, \quad dr = \frac{4}{3} r_\nu u^{1/3} du. \quad (75)$$

The spectral integral becomes

$$F_\nu \propto \nu^3 r_\nu^2 \int \frac{u^{5/3}}{e^u - 1} du. \quad (76)$$

In the intermediate frequency range

$$kT_{\text{out}} \ll h\nu \ll kT_{\text{max}}, \quad (77)$$

the limits may be approximated by 0 and ∞ , so the integral is a constant:

$$I = \int_0^\infty \frac{u^{5/3}}{e^u - 1} du. \quad (78)$$

Since $r_\nu^2 \propto \nu^{-8/3}$, equation (76) gives

$$F_\nu \propto \nu^{1/3}. \quad (79)$$

At sufficiently low frequency the whole disk is in the Rayleigh-Jeans regime and the slope approaches $F_\nu \propto \nu^2$ for finite outer radius. At high frequency the exponential Wien cutoff from the hottest annuli dominates.

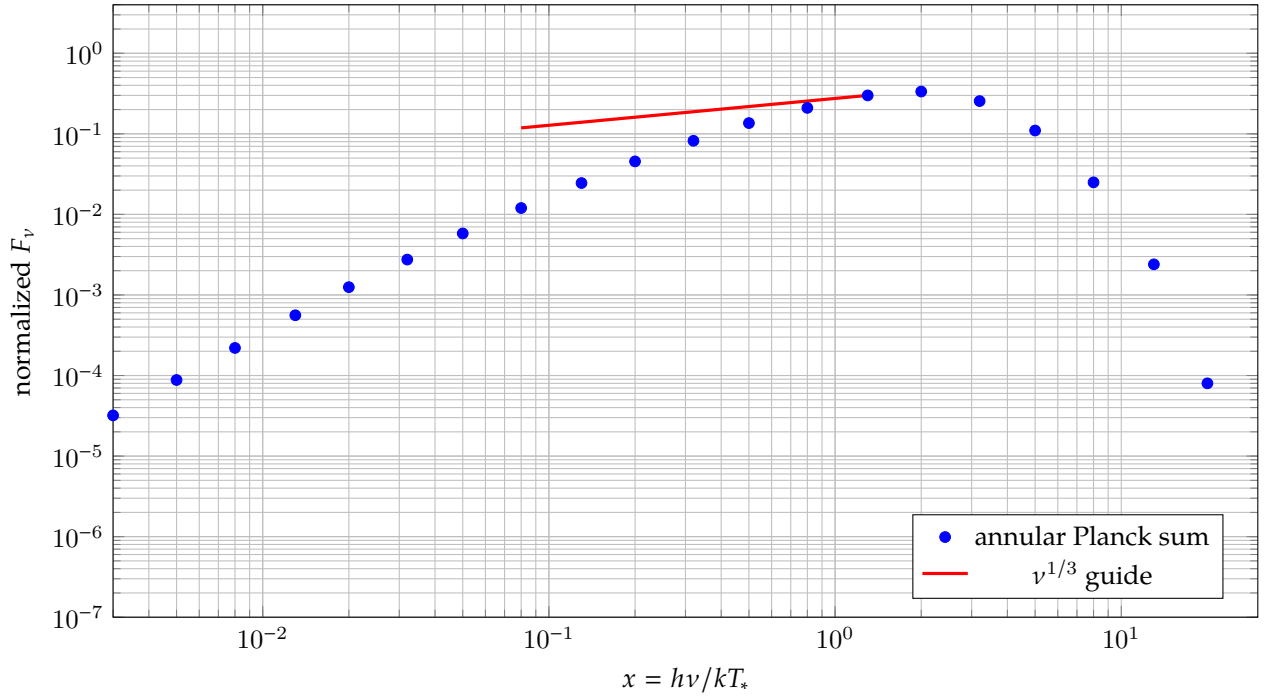


Figure 5. Dimensionless multi-colour blackbody spectrum obtained by summing Planck functions over annuli with $T(x_r) \propto x_r^{-3/4}(1 - x_r^{-1/2})^{1/4}$ between $x_r = 1.02$ and $x_r = 10^3$. The plotted coordinates are normalized to the maximum of the numerical sum. The intermediate region follows the analytic $F_v \propto \nu^{1/3}$ slope derived in equation (79).

6. The Alpha Prescription and Vertical Structure

6.1. Hydrostatic scale height

Vertical gravity near the midplane is linear in height. For $z \ll r$,

$$g_z \simeq -\Omega_K^2 z. \quad (80)$$

Hydrostatic equilibrium is

$$\frac{1}{\rho} \frac{\partial P}{\partial z} = -\Omega_K^2 z. \quad (81)$$

For an isothermal layer with $P = \rho c_s^2$,

$$c_s^2 \frac{\partial \rho}{\partial z} = -\rho \Omega_K^2 z. \quad (82)$$

Integrating gives

$$\rho(z) = \rho_0 \exp\left(-\frac{z^2}{2H^2}\right), \quad (83)$$

where

$$H = \frac{c_s}{\Omega_K}. \quad (84)$$

Since $v_K = r\Omega_K$,

$$\frac{H}{r} = \frac{c_s}{v_K}. \quad (85)$$

The thin-disk condition is therefore equivalent to orbital motion being highly supersonic:

$$\mathcal{M} = \frac{v_K}{c_s} \gg 1. \quad (86)$$

6.2. The Shakura–Sunyaev viscosity

Shakura and Sunyaev parameterized the vertically integrated stress by assuming turbulent eddies have characteristic speed below c_s and size below H . Thus

$$\nu = \alpha c_s H, \quad (87)$$

with

$$0 < \alpha \lesssim 1. \quad (88)$$

Using (84),

$$\nu = \alpha \frac{c_s^2}{\Omega_K}. \quad (89)$$

The radial drift speed follows from (27) and (45):

$$v_r = -\frac{\dot{M}}{2\pi r \Sigma}. \quad (90)$$

Eliminating $\dot{M}/\Sigma$ gives

$$v_r = -\frac{3\nu}{2r} \left[1 - \left(\frac{R_{\text{in}}}{r} \right)^{1/2} \right]^{-1}. \quad (91)$$

Far from the inner boundary,

$$|v_r| \simeq \frac{3\nu}{2r}. \quad (92)$$

With (87),

$$\frac{|v_r|}{v_K} \simeq \frac{3}{2} \alpha \left(\frac{H}{r} \right)^2. \quad (93)$$

Thus radial motion is slow because the disk is thin.

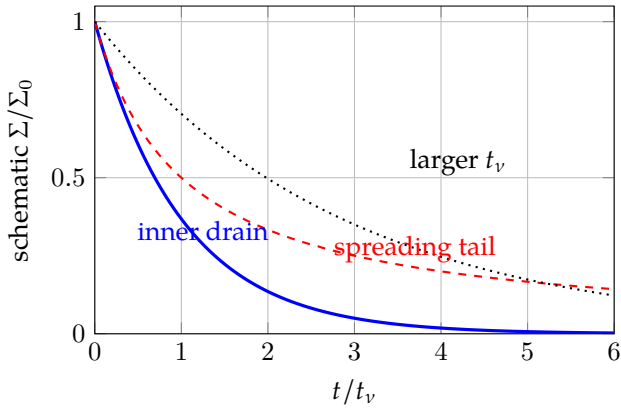


Figure 6. Schematic secular evolution of surface density governed by the viscous diffusion equation. The characteristic time is $t_v \sim r^2/\nu$, much longer than the orbital time in a thin disk.

7. Diffusion Equation and Viscous Time-scale

For a general time-dependent disk, combine the reduced angular-momentum equation with mass conservation. From (37),

$$v_r = \frac{1}{2\pi r \Sigma} \left(\frac{dj}{dr} \right)^{-1} \frac{\partial \mathcal{G}}{\partial r}. \quad (94)$$

For Keplerian rotation,

$$\frac{dj}{dr} = \frac{1}{2} (GM)^{1/2} r^{-1/2}. \quad (95)$$

The torque magnitude is

$$\mathcal{G} = 3\pi\nu\Sigma(GMr)^{1/2}. \quad (96)$$

Substitution gives

$$v_r = -\frac{3}{\Sigma r^{1/2}} \frac{\partial}{\partial r} (\nu \Sigma r^{1/2}), \quad (97)$$

where the sign corresponds to inward flow for decreasing angular-momentum flux. Inserting this result into (26) gives

$$\frac{\partial \Sigma}{\partial t} = \frac{3}{r} \frac{\partial}{\partial r} \left[r^{1/2} \frac{\partial}{\partial r} (\nu \Sigma r^{1/2}) \right]. \quad (98)$$

This nonlinear diffusion equation is the fundamental evolution equation for a thin viscous disk.

The characteristic viscous time is

$$t_v \sim \frac{r^2}{\nu}. \quad (99)$$

Using $\nu = \alpha c_s H$ and $H = c_s/\Omega_K$,

$$t_v \sim \frac{1}{\alpha} \left(\frac{r}{H} \right)^2 \Omega_K^{-1}. \quad (100)$$

The dynamical time is

$$t_{\text{dyn}} = \Omega_K^{-1}, \quad (101)$$

and the thermal time is approximately

$$t_{\text{th}} \sim \frac{1}{\alpha} \Omega_K^{-1}. \quad (102)$$

Therefore

$$t_{\text{dyn}} \ll t_{\text{th}} \ll t_v \quad (103)$$

for $\alpha < 1$ and $H/r \ll 1$. This hierarchy explains why the disk can remain vertically hydrostatic while slowly redistributing mass.

8. Radial Structure Scalings of the Shakura-Sunyaev Solution

The flux law fixes T_{eff} , but the midplane temperature, surface density, pressure, and thickness require vertical radiative transfer and an equation of state. The standard model closes the system with

$$P = P_{\text{gas}} + P_{\text{rad}}, \quad (104)$$

where

$$P_{\text{gas}} = \frac{\rho k T_c}{\mu m_p}, \quad (105)$$

and

$$P_{\text{rad}} = \frac{a T_c^4}{3}. \quad (106)$$

Radiative diffusion gives approximately

$$D = \frac{4acT_c^4}{3\kappa\Sigma}. \quad (107)$$

The surface density and midplane density are related by

$$\Sigma \simeq 2\rho_c H. \quad (108)$$

Combining (45), (87), (84), and (107) gives the classical radial zones.

Let

$$f(r) = 1 - \left(\frac{R_{\text{in}}}{r} \right)^{1/2}. \quad (109)$$

For the gas-pressure, electron-scattering region, a commonly used scaling is

$$T_c \propto \alpha^{-1/5} \dot{M}^{2/5} M^{3/10} r^{-9/10} f^{2/5}. \quad (110)$$

The corresponding surface density scales as

$$\Sigma \propto \alpha^{-4/5} \dot{M}^{3/5} M^{1/5} r^{-3/5} f^{3/5}. \quad (111)$$

The scale height behaves as

$$H \propto \alpha^{-1/10} \dot{M}^{1/5} M^{-7/20} r^{21/20} f^{1/5}. \quad (112)$$

In the gas-pressure, free-free opacity region,

$$T_c \propto \alpha^{-1/5} \dot{M}^{3/10} M^{1/4} r^{-3/4} f^{3/10}, \quad (113)$$

$$\Sigma \propto \alpha^{-4/5} \dot{M}^{7/10} M^{1/4} r^{-3/4} f^{7/10}, \quad (114)$$

and

$$H \propto \alpha^{-1/10} \dot{M}^{3/20} M^{-3/8} r^{9/8} f^{3/20}. \quad (115)$$

In the radiation-pressure, electron-scattering inner zone,

$$H \propto \dot{M} f, \quad (116)$$

while

$$\Sigma \propto \alpha^{-1} \dot{M}^{-1} M^{-1/2} r^{3/2} f^{-1}. \quad (117)$$

These proportionalities reveal a key feature: increasing $\dot{M}$ decreases Σ in the radiation-pressure zone, a fact connected to thermal and viscous instability in the simplest αP_{tot} prescription.

9. Worked Numerical Applications

9.1. Cataclysmic variable disk

Consider a white dwarf of mass

$$M = 1.0 M_{\odot} = 1.99 \times 10^{33} \text{ g} \quad (118)$$

and radius

$$R_{\text{in}} = 7.0 \times 10^8 \text{ cm}. \quad (119)$$

Let

$$\dot{M} = 10^{-9} M_{\odot} \text{ yr}^{-1} = 6.31 \times 10^{16} \text{ g s}^{-1}. \quad (120)$$

The disk luminosity is

$$L_{\text{disk}} = \frac{(6.67 \times 10^{-8})(1.99 \times 10^{33})(6.31 \times 10^{16})}{2(7.0 \times 10^8)}. \quad (121)$$

Thus

$$L_{\text{disk}} = 6.0 \times 10^{33} \text{ erg s}^{-1} \simeq 1.6 L_{\odot}. \quad (122)$$

The maximum effective temperature from (60) is

$$T_{\text{max}} \simeq 5.7 \times 10^4 \text{ K}. \quad (123)$$

This peaks in the far ultraviolet. If the white dwarf rotates slowly, a comparable luminosity may arise in the boundary layer, often at extreme ultraviolet or soft X-ray temperatures.

9.2. Stellar-mass black-hole binary

For a non-spinning black hole, take

$$R_{\text{in}} = 6 \frac{GM}{c^2}. \quad (124)$$

With

$$M = 10 M_{\odot}, \quad (125)$$

one obtains

$$R_{\text{in}} = 8.9 \times 10^6 \text{ cm}. \quad (126)$$

For

$$\dot{M} = 10^{-8} M_{\odot} \text{ yr}^{-1} = 6.31 \times 10^{17} \text{ g s}^{-1}, \quad (127)$$

the Newtonian disk luminosity is

$$L_{\text{disk}} = 4.7 \times 10^{37} \text{ erg s}^{-1}. \quad (128)$$

The Eddington luminosity is

$$L_{\text{Edd}} = 1.26 \times 10^{39} \text{ erg s}^{-1}, \quad (129)$$

so the Eddington ratio is

$$\frac{L_{\text{disk}}}{L_{\text{Edd}}} = 3.7 \times 10^{-2}. \quad (130)$$

The maximum temperature is

$$T_{\text{max}} \simeq 4.0 \times 10^6 \text{ K}, \quad (131)$$

corresponding to

$$kT_{\text{max}} \simeq 0.34 \text{ keV}. \quad (132)$$

This is the thermal soft X-ray component observed in many black-hole binaries, modified by spectral hardening due to electron scattering atmospheres.

9.3. Active galactic nucleus

For a black hole of mass

$$M = 10^8 M_{\odot} \quad (133)$$

and accretion rate

$$\dot{M} = 1 M_{\odot} \text{ yr}^{-1} = 6.31 \times 10^{25} \text{ g s}^{-1}, \quad (134)$$

the inner radius $6GM/c^2$ is

$$R_{\text{in}} = 8.9 \times 10^{13} \text{ cm}. \quad (135)$$

The disk luminosity is

$$L_{\text{disk}} = 4.7 \times 10^{45} \text{ erg s}^{-1}. \quad (136)$$

The Eddington luminosity is

$$L_{\text{Edd}} = 1.26 \times 10^{46} \text{ erg s}^{-1}, \quad (137)$$

so

$$\frac{L_{\text{disk}}}{L_{\text{Edd}}} = 0.37. \quad (138)$$

The maximum effective temperature is only

$$T_{\text{max}} \simeq 1.3 \times 10^5 \text{ K}, \quad (139)$$

placing the thermal peak in the ultraviolet. The mass dependence $T_{\text{max}} \propto M^{-1/4}$ at fixed Eddington ratio explains why stellar-mass black holes emit thermal X-rays whereas quasars peak in the optical-UV.

10. Discussion

10.1. Assumptions and limits of validity

The thin-disk equations rely on a set of coupled approximations. First, the disk must cool efficiently so that dissipated energy is radiated locally. This requires the cooling time to be shorter than the inflow time. If the cooling is inefficient, energy is advected inward and the flow becomes geometrically thick, as in advection-dominated accretion flows. Second, the disk must be optically thick for the blackbody and diffusion formulae to apply. Optically thin disks require kinetic plasma physics and nonthermal radiative processes.

Table 1. Representative scales for thin accretion disks. Values are order-of-magnitude Newtonian estimates using the equations derived in the text.

System	M	Inner radius	$\dot{M}$	L_{disk}	Characteristic T_{max}
Cataclysmic variable	$1 M_{\odot}$	$7 \times 10^8 \text{ cm}$	$10^{-9} M_{\odot} \text{ yr}^{-1}$	$6.0 \times 10^{33} \text{ erg s}^{-1}$	$5.7 \times 10^4 \text{ K}$
Stellar black hole	$10 M_{\odot}$	$9 \times 10^6 \text{ cm}$	$10^{-8} M_{\odot} \text{ yr}^{-1}$	$4.7 \times 10^{37} \text{ erg s}^{-1}$	$4.0 \times 10^6 \text{ K}$
AGN quasar disk	$10^8 M_{\odot}$	$9 \times 10^{13} \text{ cm}$	$1 M_{\odot} \text{ yr}^{-1}$	$4.7 \times 10^{45} \text{ erg s}^{-1}$	$1.3 \times 10^5 \text{ K}$

Third, the rotation law must be close to Keplerian. This is accurate when radial pressure support is small:

$$\frac{1}{\rho} \frac{\partial P}{\partial r} \ll \frac{GM}{r^2}. \quad (140)$$

Since $P \sim \rho c_s^2$, the fractional correction is

$$\frac{\Delta\Omega}{\Omega_K} \sim \left(\frac{H}{r}\right)^2. \quad (141)$$

Fourth, the zero-torque inner boundary is not universal. It is well motivated for a classical black hole if stresses vanish in the plunging region, but magnetic fields can couple material across the innermost stable circular orbit. For neutron stars and white dwarfs, the inner boundary may be a boundary layer or magnetosphere.

The Newtonian model also fails near compact relativistic objects when r is only a few gravitational radii. The relativistic disk model replaces the Newtonian energy and angular momentum by exact geodesic quantities in the Kerr metric [11, 12]. Its efficiency is

$$\eta_{\text{GR}} = 1 - E_{\text{ISCO}}, \quad (142)$$

where E_{ISCO} is the specific energy at the innermost stable circular orbit. For a Schwarzschild black hole,

$$\eta_{\text{GR}} = 0.057. \quad (143)$$

For a rapidly spinning prograde Kerr black hole it can exceed

$$\eta_{\text{GR}} \gtrsim 0.3. \quad (144)$$

10.2. Magnetorotational origin of the stress

The α prescription is not a microscopic viscosity. Molecular viscosity is far too small to explain observed accretion rates. A weakly magnetized disk with decreasing angular velocity is unstable to the magnetorotational instability. In ideal MHD, two fluid elements connected by a magnetic field line behave like masses connected by a spring. If the inner element is displaced outward and the outer inward, magnetic tension transfers angular momentum from the inner element to the outer one. The instability criterion for a weak field is

$$\frac{d\Omega^2}{dr} < 0. \quad (145)$$

Keplerian disks satisfy this condition because $\Omega^2 \propto r^{-3}$. The turbulent stress is dominated by Maxwell stress,

$$T_{r\phi}^{\text{MHD}} = \rho v_r \delta v_{\phi} - \frac{B_r B_{\phi}}{4\pi}. \quad (146)$$

The effective α is often written

$$\alpha_{\text{eff}} = \frac{\langle T_{r\phi}^{\text{MHD}} \rangle}{\langle P \rangle}. \quad (147)$$

Simulations commonly find values from 10^{-3} to 10^{-1} depending on net magnetic flux, thermodynamics, stratification, and radiation transport.

10.3. Radiation-pressure instability

In the inner regions of luminous disks, radiation pressure may dominate:

$$P_{\text{rad}} \gg P_{\text{gas}}. \quad (148)$$

If the stress scales with total pressure,

$$W_{r\phi} = \alpha P_{\text{tot}} H, \quad (149)$$

then heating responds strongly to temperature. A local estimate gives

$$Q^+ \propto P_{\text{tot}} H \Omega. \quad (150)$$

In radiation-pressure hydrostatic equilibrium, $P_{\text{rad}} \propto T_c^4$, and the heating increases more rapidly with T_c than radiative cooling:

$$Q^- \propto \frac{T_c^4}{\Sigma}. \quad (151)$$

The standard model is therefore thermally unstable in this regime. Observationally, however, many luminous disks appear more stable than the simplest analysis predicts. Possible resolutions include stress scaling with gas pressure, vertical advection, magnetic pressure support, disk winds, inhomogeneous radiation escape, and global time-dependent effects.

10.4. Comparison with observations

In cataclysmic variables, the thin-disk model explains the broad optical-UV continuum and the energy division between disk and boundary layer. Dwarf-nova outbursts require an additional thermal-viscous ionization instability, but the hot-state disk is still approximately Shakura-Sunyaev.

In X-ray binaries, the thermal dominant state is fitted with multi-colour disk models. The observed color temperature is higher than T_{eff} by a hardening factor,

$$f_{\text{col}} = \frac{T_{\text{col}}}{T_{\text{eff}}}, \quad (152)$$

typically of order 1.5–1.9. Relativistic Doppler shifts, gravitational redshift, limb darkening, and returning

radiation must be included for precision black-hole spin measurements.

In AGN, the optical–UV big blue bump is broadly consistent with optically thick accretion, yet detailed spectra often disagree with the simplest model. Observed disks can appear larger in microlensing and reverberation mapping than predicted by standard thin-disk theory. These discrepancies may point to disk winds, irradiation, inhomogeneous temperature fluctuations, or incomplete understanding of opacity and vertical structure.

10.5. Soltan-type efficiency argument

The Soltan argument compares the integrated quasar light to the local mass density in dormant black holes [16]. If the comoving radiated energy density is U_{rad} and the present black-hole mass density is ρ_{BH} , then

$$U_{\text{rad}} = \frac{\eta}{1 - \eta} \rho_{\text{BH}} c^2. \quad (153)$$

Solving for η gives

$$\eta = \frac{U_{\text{rad}}}{U_{\text{rad}} + \rho_{\text{BH}} c^2}. \quad (154)$$

Empirical estimates generally imply

$$\eta \sim 0.06\text{--}0.15, \quad (155)$$

consistent with radiatively efficient accretion onto black holes with modest spin. The argument is global and demographic, but it strongly supports the idea that quasar luminosities are powered by thin or moderately thin accretion disks rather than by stellar processes.

11. Conclusion

The Shakura–Sunyaev thin disk model is a rare example of an astrophysical theory that is both analytically transparent and observationally powerful. Beginning with mass conservation, angular-momentum conservation, and viscous shear, one obtains the steady relation

$$\nu \Sigma = \frac{\dot{M}}{3\pi} \left[1 - \left(\frac{R_{\text{in}}}{r} \right)^{1/2} \right], \quad (156)$$

and hence the local flux

$$D(r) = \frac{3GM\dot{M}}{8\pi r^3} \left[1 - \left(\frac{R_{\text{in}}}{r} \right)^{1/2} \right]. \quad (157)$$

The disk emits

$$L_{\text{disk}} = \frac{GM\dot{M}}{2R_{\text{in}}}, \quad (158)$$

while the remaining half of the Newtonian accretion energy resides as kinetic energy at the inner boundary. Local blackbody emission gives

$$T_{\text{eff}} \propto r^{-3/4} \quad (159)$$

at large radius and produces the characteristic multi-temperature continuum

$$F_{\nu} \propto \nu^{1/3} \quad (160)$$

between the outer Rayleigh–Jeans and inner Wien regimes.

The model’s central physical insight is that accretion is controlled not by the ability of matter to lose energy alone, but by the ability of matter to lose angular momentum. Viscous or turbulent stresses transport angular momentum outward, allowing mass to drift inward on the long time-scale $t_{\nu} \sim r^2/\nu$. The α prescription condenses complex turbulence into a tractable closure, and MRI-driven MHD turbulence now provides its leading physical interpretation. Nevertheless, radiation-pressure instability, relativistic inner-boundary stresses, winds, irradiation, and spectral hardening remind us that real disks are richer than the ideal theory. The thin disk remains the indispensable zeroth-order framework against which those complexities are measured.

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