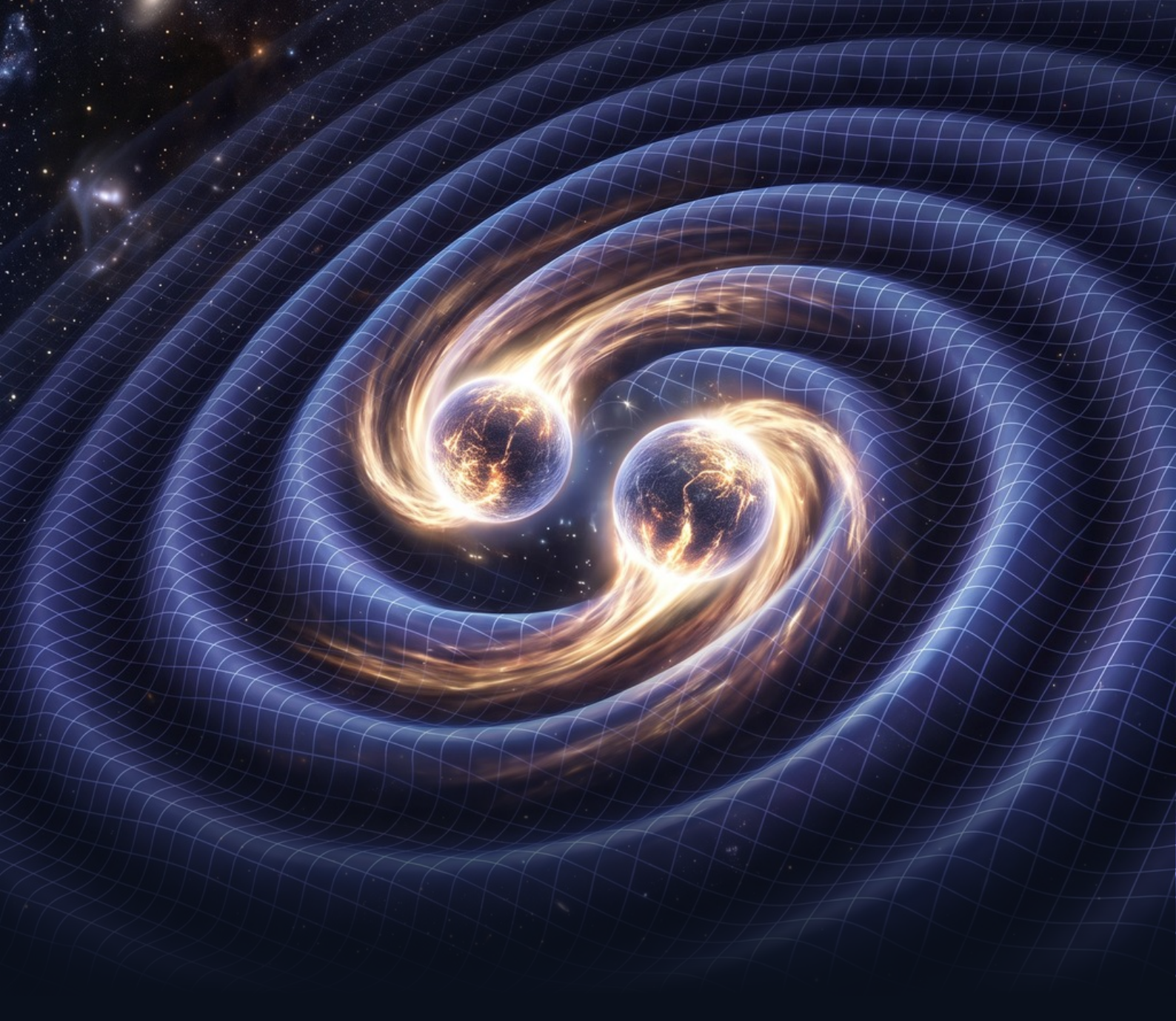




GRAVITATIONAL WAVES

Linearised Einstein Field Equations and Quadrupole Radiation Loss in Binary Systems

Academic Research Paper

Theoretical Astrophysics — Step-by-Step Analytic Derivations, Vector Diagrams and Numerical Tables

Linearised Einstein Field Equations and Quadrupole Radiation Loss in Binary Systems

Gravitational Waves

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Abstract. Gravitational radiation is the cleanest weak-field prediction of general relativity that depends on the nonlinear dynamical nature of spacetime while remaining calculable with controlled approximations. This paper develops the linearised Einstein field equations from the weak-field metric, derives the transverse-traceless plane-wave degrees of freedom, and connects them to measurable tidal strains in freely falling detectors. The retarded solution of the harmonic-gauge wave equation is expanded for slowly moving isolated sources, yielding the quadrupole waveform and the quadrupole luminosity formula. These general results are then specialised to Newtonian compact binaries. For a circular binary we derive the standard power loss, orbital decay, coalescence time, chirp mass dependence, and leading frequency evolution. Eccentric corrections are stated in Peters' form and are related to the Hulse–Taylor pulsar test. Worked numerical estimates are given for PSR B1913+16 and GW150914, including strain amplitude, chirp mass, luminosity scale, and radiated energy. The discussion emphasises assumptions, gauge issues, domain of validity, and the relationship between the elementary quadrupole treatment and modern post-Newtonian and numerical-relativity waveform modelling.

Keywords: gravitational waves; linearised gravity; quadrupole radiation; compact binaries; chirp mass; binary pulsars

1. Introduction

The direct detection of gravitational waves by laser interferometers transformed gravitational radiation from a formal consequence of Einstein's equations into an observational probe of compact objects, strong gravity, and stellar evolution. The underlying theory, however, begins in a regime that is mathematically modest: a nearly flat metric, a small perturbation, a gauge choice, and a wave equation. From these ingredients one obtains transverse waves with two tensor polarisations and a radiation formula in which the dominant source is not a mass monopole or dipole, but the second time derivative of a quadrupole moment. This is the first major conceptual distinction between gravitational and electromagnetic radiation.

The historical development was not immediate. Einstein's initial calculations of weak gravitational waves appeared in 1916 and 1918 [2, 3]; Eddington clarified aspects of coordinate artefacts in wave solutions [4]. The modern understanding of gauge freedom, energy flux, and asymptotic radiation was developed over decades and is now standard in relativistic astrophysics [5, 6, 7]. The first precision observational evidence came not from interferometers but from the orbital decay of the binary pulsar PSR B1913+16, discovered by Hulse and Taylor and monitored over many years [11, 12, 13]. The measured decrease of its orbital period agrees with the quadrupole prediction to within observational and Galactic-acceleration uncertainties. The first direct detection, GW150914, was produced by a binary black-hole merger and was announced by the LIGO Scientific Collaboration and Virgo Collaboration in 2016 [14]. It radiated roughly three solar masses of rest energy in gravitational waves over a fraction of a second.

This paper gives a self-contained derivation of the weak-field theory and its binary consequences at the level appropriate for advanced undergraduate and graduate astrophysics. The emphasis is on explicit intermediate steps and physical interpretation. We use metric signature $(-, +, +, +)$, retain factors of G and c , and denote the Minkowski metric by

$$\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1). \quad (1)$$

Greek indices run over spacetime coordinates 0, 1, 2, 3, Latin indices over spatial coordinates 1, 2, 3, and repeated indices are summed. In weak-field calculations indices on perturbations are raised and lowered with $\eta_{\mu\nu}$ unless otherwise stated.

2. Physical and Mathematical Framework

2.1. Weak fields and ordering

The Einstein field equations are

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu}. \quad (2)$$

In the weak-field regime one writes the metric as a small perturbation about flat spacetime:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad |h_{\mu\nu}| \ll 1. \quad (3)$$

The inverse metric follows from $g_{\mu\alpha}g^{\alpha\nu} = \delta_\mu^\nu$. To first order,

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + O(h^2), \quad (4)$$

because

$$(\eta_{\mu\alpha} + h_{\mu\alpha})(\eta^{\alpha\nu} - h^{\alpha\nu}) = \delta_\mu^\nu + O(h^2). \quad (5)$$

The approximation is not merely that gravity is weak; it is also that the curvature radius is long compared with the scale on which local inertial coordinates are defined. For compact binaries during early inspiral the orbital potential satisfies $GM/(ac^2) \ll 1$, while velocities obey $v/c \ll 1$. The late merger of black holes violates these conditions and requires numerical relativity.

Definition 1. The trace of the metric perturbation and its trace reverse are

$$h \equiv \eta^{\mu\nu} h_{\mu\nu}, \quad \bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h. \quad (6)$$

Taking the trace gives

$$\bar{h} = \eta^{\mu\nu} \bar{h}_{\mu\nu} = -h, \quad (7)$$

so that the inverse relation is

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \bar{h}. \quad (8)$$

The trace reverse is not a decorative notation. It is precisely the combination for which the linearised Einstein tensor simplifies in harmonic gauge.

2.2. Connection and curvature to first order

The Christoffel symbols are

$$\Gamma_{\mu\nu}^{\rho} = \frac{1}{2} g^{\rho\sigma} (\partial_{\mu} g_{\nu\sigma} + \partial_{\nu} g_{\mu\sigma} - \partial_{\sigma} g_{\mu\nu}). \quad (9)$$

Substituting (3) and (4), and discarding $O(h^2)$ terms, gives

$$\Gamma_{\mu\nu}^{\rho} = \frac{1}{2} \eta^{\rho\sigma} (\partial_{\mu} h_{\nu\sigma} + \partial_{\nu} h_{\mu\sigma} - \partial_{\sigma} h_{\mu\nu}). \quad (10)$$

The Riemann tensor is

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_{\mu} \Gamma_{\nu\sigma}^{\rho} - \partial_{\nu} \Gamma_{\mu\sigma}^{\rho} + \Gamma_{\mu\lambda}^{\rho} \Gamma_{\nu\sigma}^{\lambda} - \Gamma_{\nu\lambda}^{\rho} \Gamma_{\mu\sigma}^{\lambda}. \quad (11)$$

At linear order the quadratic products of Christoffel symbols are discarded. Lowering the first index with $\eta_{\rho\alpha}$ yields

$$R_{\rho\sigma\mu\nu} = \frac{1}{2} (\partial_{\mu} \partial_{\sigma} h_{\rho\nu} + \partial_{\nu} \partial_{\rho} h_{\sigma\mu} - \partial_{\mu} \partial_{\rho} h_{\sigma\nu} - \partial_{\nu} \partial_{\sigma} h_{\rho\mu}). \quad (12)$$

This expression is invariant under infinitesimal coordinate transformations at linear order, as expected for a curvature tensor.

Contracting the Riemann tensor gives the Ricci tensor:

$$R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu}. \quad (13)$$

Using (12),

$$R_{\mu\nu} = \frac{1}{2} (\partial_{\rho} \partial_{\mu} h^{\rho}{}_{\nu} + \partial_{\rho} \partial_{\nu} h^{\rho}{}_{\mu} - \square h_{\mu\nu} - \partial_{\mu} \partial_{\nu} h), \quad (14)$$

where the flat d'Alembertian is

$$\square \equiv \eta^{\alpha\beta} \partial_{\alpha} \partial_{\beta} = -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \nabla^2. \quad (15)$$

The Ricci scalar is

$$R = \eta^{\mu\nu} R_{\mu\nu} = \partial_{\mu} \partial_{\nu} h^{\mu\nu} - \square h. \quad (16)$$

Therefore the linearised Einstein tensor is

$$G_{\mu\nu}^{(1)} = \frac{1}{2} (\partial_{\rho} \partial_{\mu} h^{\rho}{}_{\nu} + \partial_{\rho} \partial_{\nu} h^{\rho}{}_{\mu} - \square h_{\mu\nu} - \partial_{\mu} \partial_{\nu} h - \eta_{\mu\nu} \partial_{\alpha} \partial_{\beta} h^{\alpha\beta} + \eta_{\mu\nu} \square h). \quad (17)$$

2.3. Gauge transformations

An infinitesimal coordinate transformation

$$x^{\mu} \rightarrow x'^{\mu} = x^{\mu} + \xi^{\mu}(x) \quad (18)$$

changes the perturbation according to

$$h'_{\mu\nu} = h_{\mu\nu} - \partial_{\mu} \xi_{\nu} - \partial_{\nu} \xi_{\mu}. \quad (19)$$

The trace changes as

$$h' = h - 2\partial_{\alpha} \xi^{\alpha}, \quad (20)$$

and the trace-reversed field changes as

$$\bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} - \partial_{\mu} \xi_{\nu} - \partial_{\nu} \xi_{\mu} + \eta_{\mu\nu} \partial_{\alpha} \xi^{\alpha}. \quad (21)$$

Remark 1. Gauge freedom in linearised gravity is the weak-field remnant of diffeomorphism invariance. Components of $h_{\mu\nu}$ are not individually observable. Observable effects are tidal, encoded in curvature or in gauge-invariant asymptotic wave amplitudes.

3. Linearised Einstein Equations in Lorenz Gauge

The Lorenz, or harmonic, gauge condition is

$$\partial^{\mu} \bar{h}_{\mu\nu} = 0. \quad (22)$$

The name is historically parallel to the Lorenz gauge of electromagnetism; “harmonic” refers to coordinates satisfying $\square x^{\mu} = 0$ to linear order.

Using (6), the Ricci tensor can be rewritten as

$$R_{\mu\nu} = -\frac{1}{2} \square \bar{h}_{\mu\nu} + \frac{1}{2} (\partial_{\mu} \partial^{\alpha} \bar{h}_{\alpha\nu} + \partial_{\nu} \partial^{\alpha} \bar{h}_{\alpha\mu}) - \frac{1}{4} \eta_{\mu\nu} \square \bar{h}. \quad (23)$$

The scalar curvature becomes

$$R = \partial_{\alpha} \partial_{\beta} \bar{h}^{\alpha\beta} + \frac{1}{2} \square \bar{h}. \quad (24)$$

Substitution into $G_{\mu\nu} = R_{\mu\nu} - g_{\mu\nu} R/2$ and keeping only zeroth-order $g_{\mu\nu}$ in the contraction gives

$$G_{\mu\nu}^{(1)} = -\frac{1}{2} \square \bar{h}_{\mu\nu} + \frac{1}{2} (\partial_{\mu} \partial^{\alpha} \bar{h}_{\alpha\nu} + \partial_{\nu} \partial^{\alpha} \bar{h}_{\alpha\mu})$$

$$-\frac{1}{2}\eta_{\mu\nu}\partial_\alpha\partial_\beta\bar{h}^{\alpha\beta}. \quad (25)$$

In Lorenz gauge, the last two terms vanish, leaving

$$G_{\mu\nu}^{(1)} = -\frac{1}{2}\Box\bar{h}_{\mu\nu}. \quad (26)$$

The Einstein equations (2) therefore reduce to

$$\Box\bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4}T_{\mu\nu}. \quad (27)$$

Proposition 1. *For any sufficiently smooth weak-field perturbation satisfying appropriate boundary conditions, one can impose the Lorenz gauge by a gauge transformation.*

Proof. From (21),

$$\begin{aligned} \partial^\mu\bar{h}'_{\mu\nu} &= \partial^\mu\bar{h}_{\mu\nu} - \Box\xi_\nu - \partial_\nu\partial_\mu\xi^\mu + \partial_\nu\partial_\alpha\xi^\alpha \\ &= \partial^\mu\bar{h}_{\mu\nu} - \Box\xi_\nu. \end{aligned} \quad (28)$$

Thus $\partial^\mu\bar{h}'_{\mu\nu} = 0$ is achieved by solving

$$\Box\xi_\nu = \partial^\mu\bar{h}_{\mu\nu}. \quad (29)$$

With retarded or otherwise specified boundary conditions this wave equation has a solution. $\square$

The physical content of (27) is direct: weak gravitational disturbances propagate at speed c , and the stress-energy tensor is their source. Conservation of stress energy follows consistently. Taking ∂^μ of (27) and using (22) gives

$$\partial^\mu T_{\mu\nu} = 0 \quad (30)$$

at the same order. This is the special-relativistic conservation law appropriate to matter moving in an approximately flat background.

4. Vacuum Plane Waves and the Transverse-Traceless Gauge

In vacuum, $T_{\mu\nu} = 0$, so

$$\Box\bar{h}_{\mu\nu} = 0. \quad (31)$$

A plane-wave solution is

$$\bar{h}_{\mu\nu} = A_{\mu\nu} \exp(ik_\alpha x^\alpha), \quad (32)$$

where the wave four-vector is

$$k^\mu = \left(\frac{\omega}{c}, \mathbf{k}\right), \quad k_\mu x^\mu = -\omega t + \mathbf{k} \cdot \mathbf{x}. \quad (33)$$

Substitution into (31) gives

$$k^\alpha k_\alpha = 0, \quad \omega = c|\mathbf{k}|. \quad (34)$$

The Lorenz gauge requires

$$k^\mu A_{\mu\nu} = 0. \quad (35)$$

Residual gauge transformations preserve Lorenz gauge if

$$\Box\xi^\mu = 0. \quad (36)$$

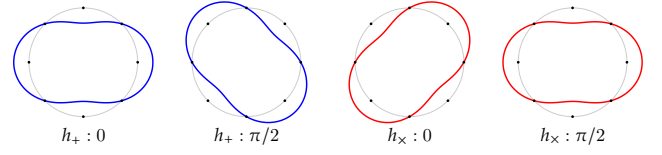


Figure 1. Tidal deformation of a ring of freely falling test masses by the two TT polarisations. The grey circle is the unperturbed ring. The plus polarisation stretches and squeezes along the x and y axes; the cross polarisation is rotated by 45° .

For a plane wave one may choose $\xi_\mu = B_\mu \exp(ik_\alpha x^\alpha)$, which changes the amplitude by

$$A'_{\mu\nu} = A_{\mu\nu} - ik_\mu B_\nu - ik_\nu B_\mu + i\eta_{\mu\nu} k_\alpha B^\alpha. \quad (37)$$

Using this residual freedom, a wave propagating in the $+z$ direction can be put in transverse-traceless (TT) gauge:

$$h_{0\mu}^{\text{TT}} = 0, \quad h_{3\mu}^{\text{TT}} = 0, \quad h^{\text{TT}i}{}_i = 0. \quad (38)$$

The nonzero spatial components are

$$h_{ij}^{\text{TT}} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij}. \quad (39)$$

For a monochromatic wave,

$$h_+ = A_+ \cos \omega \left(t - \frac{z}{c}\right), \quad (40)$$

and

$$h_\times = A_\times \cos \left[\omega \left(t - \frac{z}{c}\right) + \varphi \right]. \quad (41)$$

Equation (39) shows that gravitational waves have two independent polarisation states. They are tensor modes: under a rotation by angle ψ about the propagation direction they transform with angle 2ψ , unlike electromagnetic vector polarisations.

5. Effect on Test Masses and Detector Response

The measurable effect of a gravitational wave is tidal acceleration. Consider two neighbouring freely falling particles separated by a small spatial vector ξ^i . The geodesic deviation equation is

$$\frac{d^2\xi^\mu}{d\tau^2} = -R^\mu{}_{\alpha\nu\beta} u^\alpha u^\beta \xi^\nu. \quad (42)$$

For slowly moving test masses in a local inertial frame, $u^\mu \simeq (c, 0, 0, 0)$ and $t \simeq \tau$, so

$$\frac{d^2\xi^i}{dt^2} = -c^2 R^i{}_{0j0} \xi^j. \quad (43)$$

Using (12), and in TT gauge where $h_{00} = h_{0i} = 0$, one obtains

$$R_{i0j0} = -\frac{1}{2c^2} \frac{\partial^2 h_{ij}^{\text{TT}}}{\partial t^2}. \quad (44)$$

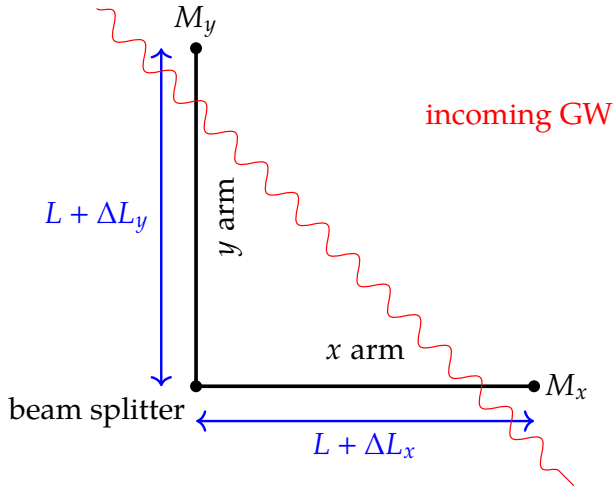


Figure 2. Idealised Michelson response. A TT gravitational wave changes the proper arm lengths with opposite signs for a normally incident plus-polarised wave. The measured signal is the differential strain in (47).

Therefore

$$\frac{d^2 \xi^i}{dt^2} = \frac{1}{2} \frac{\partial^2 h_{ij}^{\text{TT}}}{\partial t^2} \xi^j. \quad (45)$$

For an initially circular ring in the xy plane, integration to first order gives

$$\xi^i(t) = \left[\delta^i_j + \frac{1}{2} h^{\text{TT}i}_j(t) \right] \xi^j_0. \quad (46)$$

The factor $1/2$ is why a strain $h \sim 10^{-21}$ changes a 4 km interferometer arm by only $\sim 4 \times 10^{-18}$ m.

For an interferometer with orthogonal unit arm vectors $\hat{u}$ and $\hat{v}$, the differential fractional length change is

$$h_{\text{det}}(t) = \frac{1}{2} (u^i u^j - v^i v^j) h_{ij}^{\text{TT}}(t). \quad (47)$$

It is conventional to write

$$h_{\text{det}}(t) = F_+(\theta, \phi, \psi) h_+(t) + F_\times(\theta, \phi, \psi) h_\times(t), \quad (48)$$

where F_+ and $F_\times$ are antenna-pattern functions depending on sky position and polarisation angle. For arms aligned with x and y and a wave normally incident along z ,

$$F_+ = 1, \quad F_\times = 0, \quad (49)$$

if the wave polarisation axes coincide with the detector arms. A detector rotated by 45° would instead be maximally sensitive to $h_\times$.

6. Retarded Solution and Source Multipole Expansion

The retarded Green's function for the flat wave operator satisfies

$$\square G_{\text{ret}}(t, \mathbf{x}) = \delta(t) \delta^3(\mathbf{x}), \quad (50)$$

with support only in the future light cone. In three spatial dimensions,

$$G_{\text{ret}}(t, \mathbf{x}) = -\frac{1}{4\pi|\mathbf{x}|} \delta\left(t - \frac{|\mathbf{x}|}{c}\right). \quad (51)$$

Solving (27) gives

$$\bar{h}_{\mu\nu}(t, \mathbf{x}) = \frac{4G}{c^4} \int \frac{T_{\mu\nu}\left(t - \frac{|\mathbf{x} - \mathbf{x}'|}{c}, \mathbf{x}'\right)}{|\mathbf{x} - \mathbf{x}'|} d^3x'. \quad (52)$$

This is the gravitational analogue of the retarded electromagnetic potential. It is exact within the linearised approximation.

For an isolated source of size d observed at distance $r = |\mathbf{x}|$ with $r \gg d$, write $\hat{\mathbf{n}} = \mathbf{x}/r$. Then

$$|\mathbf{x} - \mathbf{x}'| = r - \hat{\mathbf{n}} \cdot \mathbf{x}' + O(d^2/r). \quad (53)$$

To leading order in d/r ,

$$\frac{1}{|\mathbf{x} - \mathbf{x}'|} \simeq \frac{1}{r}, \quad (54)$$

and the retarded time is

$$t_{\text{ret}} = t - \frac{r}{c} + \frac{\hat{\mathbf{n}} \cdot \mathbf{x}'}{c} + O(d^2/rc). \quad (55)$$

For slow internal motion, $v \ll c$, the phase variation across the source is small, so

$$\omega d/c \sim v/c \ll 1. \quad (56)$$

Then

$$\bar{h}_{ij}(t, \mathbf{x}) = \frac{4G}{c^4 r} \int T_{ij}(t - r/c, \mathbf{x}') d^3x'. \quad (57)$$

The integral of T_{ij} is related to the second derivative of the mass quadrupole. Let

$$I_{ij}(t) = \int \rho(t, \mathbf{x}) x_i x_j d^3x, \quad (58)$$

where $T^{00} \simeq \rho c^2$ in the nonrelativistic limit. Stress-energy conservation gives

$$\partial_\mu T^{\mu\nu} = 0. \quad (59)$$

For $\nu = 0$,

$$\frac{\partial T^{00}}{\partial t} + c \partial_k T^{k0} = 0, \quad (60)$$

because $x^0 = ct$. For $\nu = j$,

$$\frac{1}{c} \frac{\partial T^{0j}}{\partial t} + \partial_k T^{kj} = 0. \quad (61)$$

Taking two time derivatives of I_{ij} ,

$$\frac{d^2 I_{ij}}{dt^2} = \frac{1}{c^2} \int \frac{\partial^2 T^{00}}{\partial t^2} x_i x_j d^3x. \quad (62)$$

Using (60),

$$\frac{\partial^2 T^{00}}{\partial t^2} = -c \partial_k \frac{\partial T^{k0}}{\partial t}. \quad (63)$$

For slowly moving matter $T^{0k} = T^{k0}$, and (61) gives

$$\frac{\partial T^{k0}}{\partial t} = -c \partial_l T^{kl}. \quad (64)$$

Hence

$$\frac{\partial^2 T^{00}}{\partial t^2} = c^2 \partial_k \partial_l T^{kl}. \quad (65)$$

Substituting into (62),

$$\frac{d^2 I_{ij}}{dt^2} = \int x_i x_j \partial_k \partial_l T^{kl} d^3 x. \quad (66)$$

Integrating by parts twice and dropping surface terms for an isolated source,

$$\begin{aligned} \frac{d^2 I_{ij}}{dt^2} &= \int T^{kl} \partial_k \partial_l (x_i x_j) d^3 x \\ &= \int T^{kl} (\delta_{ki} \delta_{lj} + \delta_{kj} \delta_{li}) d^3 x \\ &= 2 \int T^{ij} d^3 x. \end{aligned} \quad (67)$$

Thus

$$\int T^{ij} d^3 x = \frac{1}{2} \frac{d^2 I_{ij}}{dt^2}. \quad (68)$$

Combining (57) and (68) gives the quadrupole waveform in harmonic gauge:

$$\bar{h}_{ij}(t, \mathbf{x}) = \frac{2G}{c^4 r} \frac{d^2 I_{ij}}{dt^2} \left(t - \frac{r}{c} \right). \quad (69)$$

The observable waveform is the TT projection:

$$h_{ij}^{\text{TT}} = \frac{2G}{c^4 r} \Lambda_{ij,kl}(\hat{\mathbf{n}}) \frac{d^2 Q_{kl}}{dt^2} \left(t - \frac{r}{c} \right). \quad (70)$$

Here

$$Q_{ij} = I_{ij} - \frac{1}{3} \delta_{ij} I_{kk} \quad (71)$$

is the reduced, trace-free quadrupole moment, and

$$\Lambda_{ij,kl} = P_{ik} P_{jl} - \frac{1}{2} P_{ij} P_{kl}, \quad P_{ij} = \delta_{ij} - n_i n_j. \quad (72)$$

Remark 2. Conservation eliminates monopole and dipole radiation. The monopole is the total mass-energy, conserved for an isolated slow source. The mass dipole is the centre of mass, whose second derivative vanishes in uniform motion. The quadrupole is therefore the leading radiative mass multipole.

7. Energy Flux and Quadrupole Luminosity

Although gravitational energy cannot be localised in a coordinate-independent tensor, weak waves in the radiation zone possess a gauge-invariant averaged flux. For a TT wave the effective energy flux in direction $\hat{\mathbf{n}}$ is

$$\frac{dE}{dA dt} = \frac{c^3}{32\pi G} \left\langle \dot{h}_{ij}^{\text{TT}} \dot{h}_{ij}^{\text{TT}} \right\rangle. \quad (73)$$

The angular brackets denote an average over several wavelengths or periods. Substituting the TT quadrupole waveform and integrating over solid angle yields

$$P \equiv -\frac{dE}{dt} = \frac{G}{5c^5} \left\langle \frac{d^3 Q_{ij}}{dt^3} \frac{d^3 Q_{ij}}{dt^3} \right\rangle. \quad (74)$$

A key angular identity used in the derivation is

$$\int n_i n_j n_k n_l d\Omega = \frac{4\pi}{15} (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}). \quad (75)$$

Equation (74) is the gravitational quadrupole luminosity formula. It contains the small coefficient G/c^5 , explaining why laboratory gravitational radiation is negligible, while compact relativistic binaries can be luminous.

8. Circular Binary: Quadrupole, Power, and Inspiral

8.1. Geometry and quadrupole tensor

Consider two point masses m_1 and m_2 in a Newtonian circular orbit of separation a . Define

$$M = m_1 + m_2, \quad \mu = \frac{m_1 m_2}{M}. \quad (76)$$

The relative separation vector is

$$\mathbf{x}(t) = a(\cos \Omega t, \sin \Omega t, 0), \quad (77)$$

with Keplerian angular frequency

$$\Omega^2 = \frac{GM}{a^3}. \quad (78)$$

The positions relative to the centre of mass are

$$\mathbf{x}_1 = \frac{m_2}{M} \mathbf{x}, \quad \mathbf{x}_2 = -\frac{m_1}{M} \mathbf{x}. \quad (79)$$

The mass quadrupole becomes

$$\begin{aligned} I_{ij} &= m_1 x_{1i} x_{1j} + m_2 x_{2i} x_{2j} \\ &= \mu x_i x_j. \end{aligned} \quad (80)$$

Its trace is

$$I_{kk} = \mu a^2. \quad (81)$$

Therefore

$$Q_{ij} = \mu \left(x_i x_j - \frac{1}{3} \delta_{ij} a^2 \right). \quad (82)$$

Explicitly,

$$Q_{xx} = \mu a^2 \left(\frac{1}{6} + \frac{1}{2} \cos 2\Omega t \right), \quad (83)$$

$$Q_{yy} = \mu a^2 \left(\frac{1}{6} - \frac{1}{2} \cos 2\Omega t \right), \quad (84)$$

$$Q_{xy} = \frac{1}{2} \mu a^2 \sin 2\Omega t, \quad (85)$$

$$Q_{zz} = -\frac{1}{3} \mu a^2. \quad (86)$$

The constant pieces do not radiate. Taking three time derivatives,

$$\ddot{Q}_{xx} = 4\mu a^2 \Omega^3 \sin 2\Omega t, \quad (87)$$

$$\ddot{Q}_{yy} = -4\mu a^2 \Omega^3 \sin 2\Omega t, \quad (88)$$

$$\ddot{Q}_{xy} = -4\mu a^2 \Omega^3 \cos 2\Omega t. \quad (89)$$

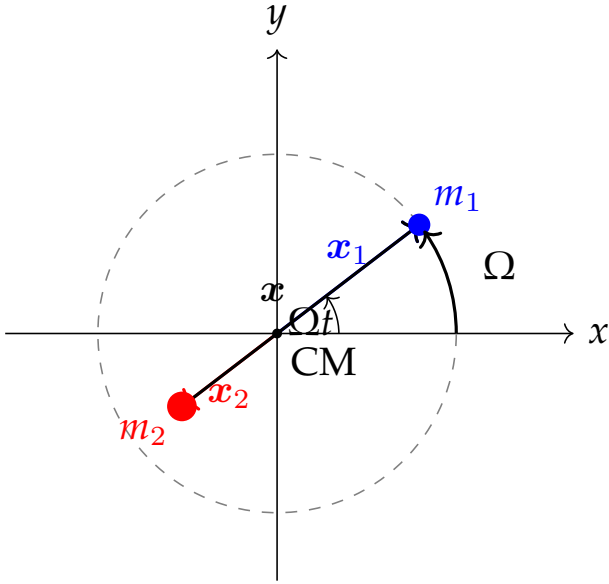


Figure 3. Circular binary in centre-of-mass coordinates. The relative vector $\mathbf{x}$ has magnitude a and rotates with angular frequency $\Omega = (GM/a^3)^{1/2}$.

Since $Q_{xy} = Q_{yx}$,

$$\begin{aligned} \ddot{Q}_{ij}\ddot{Q}_{ij} &= (\ddot{Q}_{xx})^2 + (\ddot{Q}_{yy})^2 + 2(\ddot{Q}_{xy})^2 \\ &= 32\mu^2 a^4 \Omega^6. \end{aligned} \quad (90)$$

Substitution into (74) gives

$$P = \frac{32}{5} \frac{G}{c^5} \mu^2 a^4 \Omega^6. \quad (91)$$

Using Kepler's law,

$$a^4 \Omega^6 = \frac{G^3 M^3}{a^5}, \quad (92)$$

so

$$P = \frac{32}{5} \frac{G^4}{c^5} \frac{m_1^2 m_2^2 (m_1 + m_2)}{a^5}. \quad (93)$$

8.2. Orbital decay and coalescence time

The Newtonian orbital binding energy of a circular binary is

$$E = -\frac{Gm_1 m_2}{2a} = -\frac{G\mu M}{2a}. \quad (94)$$

Energy balance gives

$$\frac{dE}{dt} = -P. \quad (95)$$

Since

$$\frac{dE}{da} = \frac{G\mu M}{2a^2}, \quad (96)$$

we have

$$\frac{da}{dt} = -\frac{2a^2 P}{G\mu M}. \quad (97)$$

Using (93),

$$\frac{da}{dt} = -\frac{64}{5} \frac{G^3 m_1 m_2 (m_1 + m_2)}{c^5 a^3}. \quad (98)$$

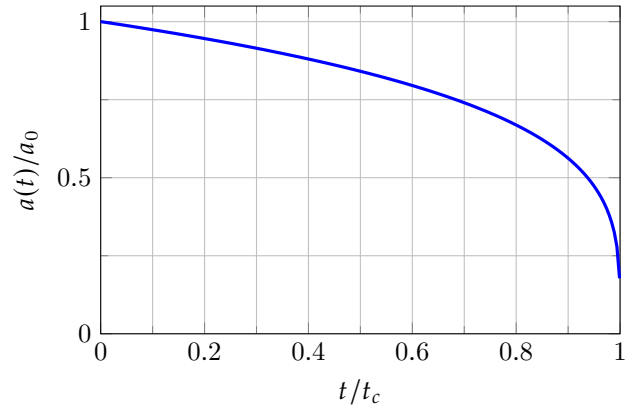


Figure 4. Leading Newtonian inspiral separation $a(t) = a_0(1 - t/t_c)^{1/4}$. The final divergence of da/dt signals breakdown of the Newtonian quadrupole approximation before actual merger.

This differential equation integrates directly:

$$a^3 da = -\frac{64}{5} \frac{G^3 m_1 m_2 M}{c^5} dt. \quad (99)$$

With $a(0) = a_0$,

$$a^4(t) = a_0^4 - \frac{256}{5} \frac{G^3 m_1 m_2 M}{c^5} t. \quad (100)$$

The coalescence time in this leading approximation is

$$t_c = \frac{5}{256} \frac{c^5 a_0^4}{G^3 m_1 m_2 (m_1 + m_2)}. \quad (101)$$

Equivalently,

$$a(t) = a_0 \left(1 - \frac{t}{t_c}\right)^{1/4}. \quad (102)$$

9. Eccentric Orbits and Peters Enhancement

Real compact binaries may have eccentricity. Peters and Mathews derived orbit-averaged gravitational-wave losses for Keplerian ellipses [9, 10]. For semi-major axis a and eccentricity e , the averaged power is

$$\langle P \rangle_e = \frac{32}{5} \frac{G^4}{c^5} \frac{m_1^2 m_2^2 M}{a^5} f(e), \quad (103)$$

where the eccentricity enhancement factor is

$$f(e) = \frac{1 + \frac{73}{24}e^2 + \frac{37}{96}e^4}{(1 - e^2)^{7/2}}. \quad (104)$$

The semi-major axis and eccentricity evolve as

$$\left\langle \frac{da}{dt} \right\rangle = -\frac{64}{5} \frac{G^3 m_1 m_2 M}{c^5 a^3 (1 - e^2)^{7/2}} \left(1 + \frac{73}{24}e^2 + \frac{37}{96}e^4\right), \quad (105)$$

and

$$\left\langle \frac{de}{dt} \right\rangle = -\frac{304}{15} e \frac{G^3 m_1 m_2 M}{c^5 a^4 (1 - e^2)^{5/2}} \left(1 + \frac{121}{304}e^2\right). \quad (106)$$

The factor $f(e)$ becomes large as $e \rightarrow 1$, because radiation is concentrated near periastron where velocities and accelerations are greatest. Equation (106) shows that gravitational radiation circularises bound binaries in isolation.

10. Chirp Mass, Frequency Evolution, and Strain

The gravitational-wave frequency for the dominant quadrupole harmonic of a circular binary is twice the orbital frequency:

$$f = \frac{\Omega}{\pi}. \quad (107)$$

Combining (78) and (107),

$$a = \left[\frac{GM}{(\pi f)^2} \right]^{1/3}. \quad (108)$$

The chirp mass is defined by

$$\mathcal{M} = \mu^{3/5} M^{2/5} = \frac{(m_1 m_2)^{3/5}}{(m_1 + m_2)^{1/5}}. \quad (109)$$

It is the best-measured mass combination in the early inspiral because it controls the leading frequency derivative.

Differentiate (107) with respect to a :

$$f = \frac{1}{\pi} \left(\frac{GM}{a^3} \right)^{1/2}, \quad \frac{df}{da} = -\frac{3}{2} \frac{f}{a}. \quad (110)$$

Using (98),

$$\begin{aligned} \frac{df}{dt} &= \frac{df}{da} \frac{da}{dt} \\ &= \frac{96}{5} \frac{G^3 m_1 m_2 M}{c^5 a^4} f. \end{aligned} \quad (111)$$

Substituting (108) gives

$$\frac{df}{dt} = \frac{96}{5} \pi^{8/3} \left(\frac{GM}{c^3} \right)^{5/3} f^{11/3}. \quad (112)$$

Thus the chirp accelerates rapidly: the time spent at high frequency is short. Integrating (112),

$$t_c - t = \frac{5}{256} \left(\frac{GM}{c^3} \right)^{-5/3} (\pi f)^{-8/3}. \quad (113)$$

Solving for $f(t)$,

$$f(t) = \frac{1}{\pi} \left[\frac{5}{256(t_c - t)} \right]^{3/8} \left(\frac{GM}{c^3} \right)^{-5/8}. \quad (114)$$

The phase satisfies

$$\phi(t) = 2\pi \int^t f(t') dt', \quad (115)$$

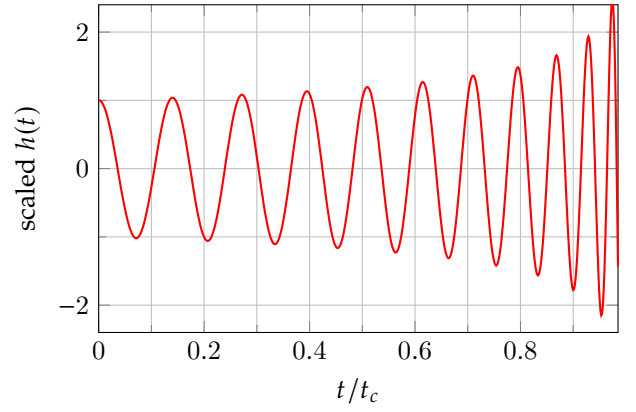


Figure 5. Analytic Newtonian chirp model with $h(t) \propto f(t)^{2/3} \cos \phi(t)$. The amplitude grows as $(t_c - t)^{-1/4}$ and the instantaneous frequency grows as $(t_c - t)^{-3/8}$ until the approximation fails near merger.

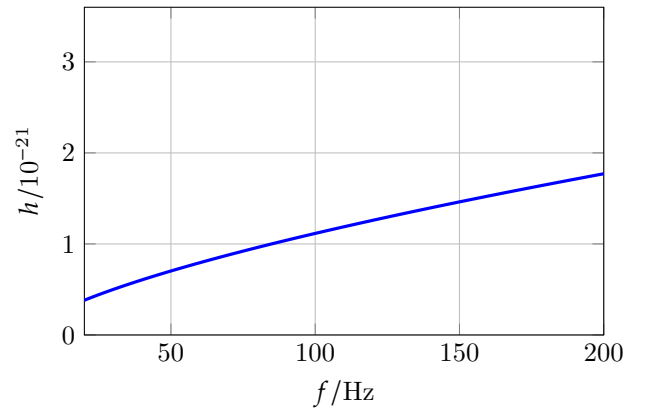


Figure 6. Leading strain scaling $h \propto f^{2/3}$ for a fixed chirp mass and distance, normalised to $h = 0.5 \times 10^{-21}$ at 30 Hz. The final observed waveform requires post-Newtonian, merger, and ringdown corrections.

so, up to an arbitrary constant,

$$\phi(t) = \phi_c - 2 \left[\frac{c^3(t_c - t)}{5GM} \right]^{5/8}. \quad (116)$$

For a circular binary at luminosity distance r , the leading strain amplitude is of order

$$h \simeq \frac{4}{r} \left(\frac{GM}{c^2} \right)^{5/3} \left(\frac{\pi f}{c} \right)^{2/3}, \quad (117)$$

for a favourably oriented source. More detailed expressions include inclination ι :

$$h_+(t) = \frac{4G^{5/3}\mathcal{M}^{5/3}}{c^4 r} (\pi f)^{2/3} \frac{1 + \cos^2 \iota}{2} \cos \phi(t), \quad (118)$$

$$h_\times(t) = \frac{4G^{5/3}\mathcal{M}^{5/3}}{c^4 r} (\pi f)^{2/3} \cos \iota \sin \phi(t). \quad (119)$$

Table 1. Representative numerical scales for gravitational-wave binaries. Values are rounded and intended for physical orientation rather than parameter estimation.

Quantity	PSR B1913+16	GW150914 scale
m_1	$1.44 M_\odot$	$36 M_\odot$
m_2	$1.39 M_\odot$	$29 M_\odot$
$\mathcal{M}$	$1.23 M_\odot$	$28 M_\odot$
Orbital period	7.75 h	milliseconds
Eccentricity	0.617	small near band
Observed ratio	0.997	direct strain
Peak strain	not direct	$\sim 10^{-21}$
Radiated energy	secular loss	$\sim 3 M_\odot c^2$

11. Worked Numerical Applications

11.1. Useful constants

We use

$$G = 6.67430 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}, \quad (120)$$

$$c = 2.99792458 \times 10^8 \text{ m s}^{-1}, \quad (121)$$

and

$$M_\odot = 1.98847 \times 10^{30} \text{ kg}. \quad (122)$$

The solar gravitational time is

$$\frac{GM_\odot}{c^3} = 4.9255 \times 10^{-6} \text{ s}, \quad (123)$$

and the solar gravitational radius is

$$\frac{GM_\odot}{c^2} = 1.477 \text{ km}. \quad (124)$$

11.2. Hulse–Taylor binary pulsar

PSR B1913+16 has masses approximately

$$m_1 \simeq 1.44 M_\odot, \quad m_2 \simeq 1.39 M_\odot, \quad (125)$$

orbital period

$$P_b \simeq 7.75 \text{ h} = 2.79 \times 10^4 \text{ s}, \quad (126)$$

and eccentricity

$$e \simeq 0.617. \quad (127)$$

The semi-major axis follows from Kepler’s law:

$$a^3 = GM \left(\frac{P_b}{2\pi} \right)^2. \quad (128)$$

With $M \simeq 2.83 M_\odot$, this gives

$$a \simeq 1.95 \times 10^9 \text{ m}. \quad (129)$$

The eccentricity enhancement is

$$f(0.617) = \frac{1 + \frac{73}{24}(0.617)^2 + \frac{37}{96}(0.617)^4}{[1 - (0.617)^2]^{7/2}}$$

$$\simeq 11.8. \quad (130)$$

Thus the luminosity is much larger than a circular binary with the same a . Long-term timing measures the orbital period derivative after correcting for kinematic and Galactic effects. The observed-to-predicted ratio is approximately

$$\frac{\dot{P}_{b,\text{obs}}}{\dot{P}_{b,\text{GR}}} \simeq 0.997, \quad (131)$$

with the remaining difference consistent with systematic uncertainties [13]. This was the first high-precision evidence that gravitational waves carry energy exactly as predicted by general relativity.

11.3. A circular compact binary timescale

For an equal-mass neutron-star binary with

$$m_1 = m_2 = 1.4 M_\odot, \quad a_0 = 10^9 \text{ m}, \quad (132)$$

the circular coalescence time from (101) is

$$t_c = \frac{5}{256} \frac{c^5 a_0^4}{G^3 m_1 m_2 M} \simeq 2.7 \times 10^{14} \text{ s} \simeq 8.6 \times 10^6 \text{ yr}. \quad (133)$$

This illustrates the steep a_0^4 dependence: reducing the initial separation by a factor of 10 reduces the inspiral time by 10^4 .

11.4. GW150914 order-of-magnitude estimates

GW150914 was consistent with source-frame component masses roughly

$$m_1 \simeq 36 M_\odot, \quad m_2 \simeq 29 M_\odot. \quad (134)$$

The chirp mass is

$$\mathcal{M} = \frac{(36 M_\odot 29 M_\odot)^{3/5}}{(65 M_\odot)^{1/5}} \simeq 28 M_\odot, \quad (135)$$

often rounded to “about $30 M_\odot$ ” for estimates. Taking

$$f \simeq 150 \text{ Hz}, \quad r \simeq 410 \text{ Mpc}, \quad (136)$$

and

$$410 \text{ Mpc} \simeq 1.27 \times 10^{25} \text{ m}, \quad (137)$$

the leading strain estimate (117) gives

$$h \sim \frac{4}{r} \left(\frac{GM}{c^2} \right)^{5/3} \left(\frac{\pi f}{c} \right)^{2/3} \sim 10^{-21}. \quad (138)$$

The total radiated energy was approximately

$$E_{\text{GW}} \simeq 3 M_\odot c^2. \quad (139)$$

Numerically,

$$3 M_\odot c^2 = 3(1.98847 \times 10^{30} \text{ kg})(2.9979 \times 10^8 \text{ m s}^{-1})^2$$

$$\simeq 5.36 \times 10^{47} \text{ J.} \quad (140)$$

If emitted over $\Delta t \sim 0.1 \text{ s}$, the characteristic luminosity is

$$L_{\text{GW}} \sim 5 \times 10^{48} \text{ W,} \quad (141)$$

vastly exceeding the electromagnetic luminosity of the observable Universe for a brief instant. This comparison is not paradoxical: gravitational waves couple weakly and escape almost freely.

12. Discussion

12.1. Assumptions and limits of validity

The derivations above rest on several assumptions. First, the field is weak:

$$|h_{\mu\nu}| \ll 1. \quad (142)$$

This condition is excellent at Earth for all detected gravitational waves, where $h \sim 10^{-21}$, but not near the horizons of merging black holes. Second, the source multipole expansion assumes slow motion:

$$\frac{v}{c} \ll 1, \quad \frac{GM}{ac^2} \ll 1. \quad (143)$$

For a circular binary these are related by

$$\frac{v^2}{c^2} \simeq \frac{GM}{ac^2}. \quad (144)$$

The Newtonian quadrupole formula is therefore the leading term in an expansion in v/c , now systematised as post-Newtonian theory [15]. Near merger, v/c is no longer small and numerical relativity is required. Third, the radiation-reaction treatment assumes adiabatic inspiral: the orbital elements change little in one orbit. This requires

$$\left| \frac{\dot{a}}{a} \right| \ll \Omega. \quad (145)$$

The condition fails near coalescence, as suggested by the formal divergence of the Newtonian chirp amplitude in Fig. 5.

12.2. Gauge and observables

A common conceptual difficulty is that gravitational waves are described by metric perturbations, while metric components are coordinate dependent. The resolution is that the measurable quantities are curvature, proper-time differences, and asymptotic TT amplitudes. In a detector small compared with the wavelength, geodesic deviation (45) is the clean local statement. In a laser interferometer whose arms are not infinitesimal relative to the wavelength, the full detector response includes light propagation along null geodesics, but (47) captures the leading long-wavelength limit.

The TT gauge is especially transparent in vacuum, but it is not generally a convenient global gauge inside

matter. The harmonic-gauge field $\bar{h}_{\mu\nu}$ is useful for solving the sourced wave equation, while the TT projection extracts the radiative degrees of freedom at infinity. The transition from (69) to (70) is therefore not optional: detectors respond to transverse trace-free tidal fields, not to arbitrary components of $\bar{h}_{\mu\nu}$.

12.3. Comparison with observations

The Hulse–Taylor pulsar tests the orbit-averaged, weak-radiation-reaction regime. Its orbital velocities are relativistic by everyday standards but still modest compared with c , and the system completes many orbits while its elements evolve slowly. The agreement ratio near 0.997 in (131) is one of the cleanest confirmations of the quadrupole formula after accounting for Galactic acceleration and proper motion.

Ground-based interferometers test a complementary regime. The early inspiral of GW150914 is reasonably described by chirp-mass scaling, but the observed signal also contains late inspiral, merger, and ringdown. The full waveform used in data analysis is not merely (118). It includes higher post-Newtonian corrections, spin effects, precession when relevant, calibration of merger against numerical relativity, and black-hole quasi-normal ringing. Nevertheless, the simple estimate (138) explains why kilometre-scale interferometers with strain sensitivity near 10^{-21} are appropriate instruments.

12.4. Modern developments

Modern gravitational-wave astronomy extends the elementary theory in several directions. Matched-filter searches require waveform phase accuracy over many cycles; since a phase error of order one radian can reduce detectability, high-order post-Newtonian expansions and effective-one-body methods are essential. Numerical relativity solves the full Einstein equations for the nonlinear merger and ringdown. Bayesian parameter estimation infers masses, spins, distances, inclinations, sky locations, and tests of general relativity from detector networks.

The quadrupole formula remains central despite these developments. It gives the leading-order amplitude, luminosity, orbital decay, chirp mass, and qualitative behaviour of nearly every compact-binary inspiral. It also explains why binaries dominate the gravitational-wave sky: they possess large, rapidly varying quadrupole moments. Spherically symmetric collapse emits no gravitational waves in general relativity, and axisymmetric systems radiate less efficiently than strongly nonspherical compact binaries.

13. Conclusion

Linearised general relativity turns Einstein's nonlinear field equations into a hyperbolic wave equation for the trace-reversed perturbation in Lorenz gauge. Vacuum

solutions contain two transverse-traceless tensor polarisations, h_+ and $h_\times$, whose physical action is tidal deformation of freely falling test masses. The retarded solution for a slow isolated source leads, after stress-energy conservation and a far-zone expansion, to the quadrupole waveform

$$\bar{h}_{ij} = \frac{2G}{c^4 r} \ddot{I}_{ij}, \quad (146)$$

and the luminosity

$$P = \frac{G}{5c^5} \langle \ddot{Q}_{ij} \ddot{Q}_{ij} \rangle. \quad (147)$$

For a circular binary this gives

$$P = \frac{32}{5} \frac{G^4}{c^5} \frac{m_1^2 m_2^2 M}{a^5}, \quad (148)$$

and

$$\dot{a} = -\frac{64}{5} \frac{G^3 m_1 m_2 M}{c^5 a^3}. \quad (149)$$

The same framework yields the chirp relation

$$\dot{f} \propto \mathcal{M}^{5/3} f^{11/3}, \quad (150)$$

which is the foundation of compact-binary mass measurement. The Hulse–Taylor pulsar verifies the secular energy-loss prediction, while GW150914 demonstrates direct detection of the strain and the enormous luminosity of black-hole coalescence. Thus the weak-field quadrupole theory is both a rigorous limiting case of general relativity and the conceptual gateway to modern gravitational-wave astrophysics.

References

- [1] B. W. Carroll and D. A. Ostlie, *An Introduction to Modern Astrophysics*, 2nd ed., Addison-Wesley, San Francisco (2007).
- [2] A. Einstein, “Näherungsweise Integration der Feldgleichungen der Gravitation,” *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften* (1916), 688–696.
- [3] A. Einstein, “Über Gravitationswellen,” *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften* (1918), 154–167.
- [4] A. S. Eddington, “The propagation of gravitational waves,” *Proceedings of the Royal Society of London A* **102** (1922), 268–282.
- [5] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation*, W. H. Freeman, San Francisco (1973).
- [6] S. Weinberg, *Gravitation and Cosmology*, Wiley, New York (1972).
- [7] M. Maggiore, *Gravitational Waves, Volume 1: Theory and Experiments*, Oxford University Press, Oxford (2008).
- [8] B. F. Schutz, *A First Course in General Relativity*, 2nd ed., Cambridge University Press, Cambridge (2009).
- [9] P. C. Peters and J. Mathews, “Gravitational radiation from point masses in a Keplerian orbit,” *Physical Review* **131** (1963), 435–440.
- [10] P. C. Peters, “Gravitational radiation and the motion of two point masses,” *Physical Review* **136** (1964), B1224–B1232.
- [11] R. A. Hulse and J. H. Taylor, “Discovery of a pulsar in a binary system,” *Astrophysical Journal* **195** (1975), L51–L53.
- [12] J. H. Taylor and J. M. Weisberg, “Further experimental tests of relativistic gravity using the binary pulsar PSR 1913+16,” *Astrophysical Journal* **345** (1989), 434–450.
- [13] J. M. Weisberg and Y. Huang, “Relativistic measurements from timing the binary pulsar PSR B1913+16,” *Astrophysical Journal* **829** (2016), 55.
- [14] B. P. Abbott et al. (LIGO Scientific Collaboration and Virgo Collaboration), “Observation of gravitational waves from a binary black hole merger,” *Physical Review Letters* **116** (2016), 061102.
- [15] L. Blanchet, “Gravitational radiation from post-Newtonian sources and inspiralling compact binaries,” *Living Reviews in Relativity* **17** (2014), 2.