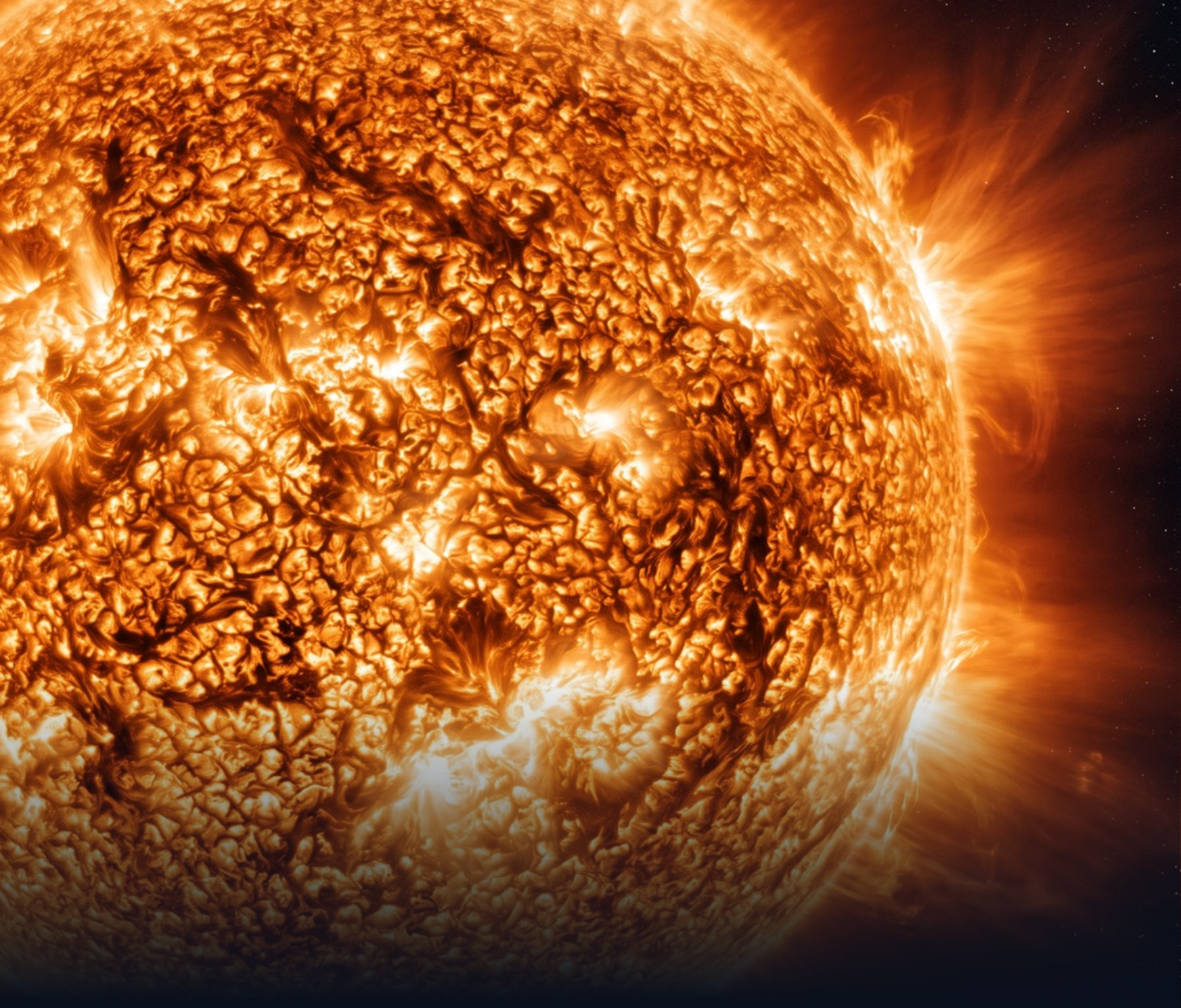




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RADIATIVE TRANSFER

# **Analytic Derivation of the Saha-Boltzmann Equations, the Grey Atmosphere Model and the Eddington Approximation**

Academic Research Paper

Theoretical Astrophysics — Step-by-Step Analytic Derivations, Vector Diagrams and Numerical Tables

# Analytic Derivation of the Saha-Boltzmann Equations, the Grey Atmosphere Model and the Eddington Approximation

Radiative Transfer

September 2026

**Abstract.** Radiative transfer connects microscopic atomic physics to the emergent spectra and colours of stars. This paper gives a self-contained analytic derivation of the Saha–Boltzmann equations, the grey atmosphere model, and the Eddington approximation, emphasizing the assumptions under which each result is valid. We begin with the angular moments of the specific intensity and derive the relations among mean intensity, flux, and radiation pressure. The Boltzmann excitation formula is obtained from statistical weights and the partition function, and the Saha ionization equation is then derived by applying the same equilibrium counting argument to the continuum phase space of free electrons. We apply the result to hydrogen at fixed electron pressure and show why Balmer absorption is strongest near A-star temperatures. We then derive the transfer equation, optical depth, formal solution, moment equations for a plane-parallel atmosphere, the grey radiative-equilibrium solution, and the Eddington temperature law. Additional derivations clarify contribution functions, the Eddington–Barbier relation, diffusion transport, and the relation between optical depth and gas pressure. Worked numerical examples and observational tests illustrate both the utility and the limitations of the classical grey atmosphere.

**Keywords:** radiative transfer; stellar atmospheres; Saha equation; Boltzmann excitation; grey atmosphere; Eddington approximation

## 1. Introduction

The visible surface of a star is not a material boundary. It is the region from which photons have a probability of order unity to escape without further interaction. The spectrum that reaches an observer is therefore a nonlocal diagnostic of temperature, density, ionization balance, opacity, and velocity over a finite optical-depth range. The analytic theory of stellar atmospheres was built to make this statement quantitative. Its classical ingredients are Kirchhoff’s law of thermal emission, Planck’s radiation law, the Boltzmann distribution of bound levels, Saha’s theory of thermal ionization, and the moment description of radiative transfer developed by Schwarzschild, Eddington, Milne, and Chandrasekhar [1, 2, 4, 6, 5, 7].

The essential physical difficulty is that the radiation field is directional. Unlike a gas in local thermodynamic equilibrium, for which the Maxwell–Boltzmann distribution is usually isotropic, photons in an atmosphere stream preferentially outward. The local matter may be close to thermodynamic equilibrium, so that its emissivity and absorptivity are determined by a temperature, but the radiation field itself is generally not a blackbody field. The specific intensity  $I_\nu$  is consequently the central quantity: it measures energy per unit area, time, frequency, and solid angle, and it preserves directional information. Its angular moments define the mean intensity  $J_\nu$ , radiative flux  $F_\nu$ , and radiation pressure  $P_\nu$ .

The first half of this paper concerns the thermodynamic state of the gas. The Boltzmann equation determines the relative populations of bound atomic levels. The Saha equation determines the ratio of successive ionization stages by applying the same statistical-

mechanical logic to a continuum of free-electron states. Their combination explains spectral classification: a strong line requires both a significant abundance of the relevant ion and a significant population in the lower level of the transition. Hydrogen Balmer lines, for example, are weak in cool stars because few atoms occupy the  $n = 2$  level, and weak in hot stars because hydrogen is ionized. Their maximum strength near  $T \approx 9500\text{--}10500\text{ K}$  follows naturally from the competition between excitation and ionization [11, 12].

The second half concerns the escape of radiation. The transfer equation along a ray balances extinction and emission. When written in optical-depth coordinates it admits a formal integral solution that exposes the meaning of optical depth as a photon survival variable. In a plane-parallel atmosphere the transfer equation can be integrated over angle to produce moment equations. With grey opacity, radiative equilibrium, and the Eddington closure  $K = J/3$ , those moment equations yield the classical grey temperature structure

$$T^4(\tau) = \frac{3}{4} T_{\text{eff}}^4 \left( \tau + \frac{2}{3} \right). \quad (1)$$

This result is both remarkably useful and profoundly approximate. It gives the standard photospheric condition  $\tau = 2/3$ , predicts a simple limb-darkening law, and provides a first analytic atmosphere for stellar-structure boundary conditions. Yet real stellar atmospheres are non-grey, frequency-dependent, convective in many layers, and often affected by departures from local thermodynamic equilibrium. Modern atmosphere codes solve the frequency- and angle-dependent transfer problem numerically [8, 15], but the analytic grey solution remains the conceptual backbone on which those calculations are understood.

## 2. Physical and Mathematical Framework

**Definition 1** (Specific intensity). The monochromatic specific intensity  $I_\nu(\mathbf{x}, \hat{\mathbf{n}}, t)$  is defined so that the energy crossing an oriented area element  $dA$  in time  $dt$ , frequency interval  $d\nu$ , and solid angle  $d\Omega$  about direction  $\hat{\mathbf{n}}$  is

$$dE_\nu = I_\nu \cos \theta dA dt d\nu d\Omega, \quad (2)$$

where  $\theta$  is the angle between  $\hat{\mathbf{n}}$  and the surface normal.

The dimensions are

$$[I_\nu] = \text{erg s}^{-1} \text{cm}^{-2} \text{Hz}^{-1} \text{sr}^{-1}. \quad (3)$$

The defining property of intensity is geometrical invariance in vacuum: in Euclidean space, without absorption or emission,  $I_\nu$  is constant along a ray. Geometrical dilution of flux arises not from a decrease in intensity but from the shrinking solid angle subtended by the source.

For an atmosphere with a preferred vertical direction, write

$$\mu = \cos \theta. \quad (4)$$

The zeroth, first, and second angular moments are defined by

$$J_\nu = \frac{1}{4\pi} \int I_\nu d\Omega, \quad (5)$$

$$H_\nu = \frac{1}{4\pi} \int \mu I_\nu d\Omega, \quad (6)$$

$$K_\nu = \frac{1}{4\pi} \int \mu^2 I_\nu d\Omega. \quad (7)$$

The corresponding physical flux and radiation pressure are

$$F_\nu = \int \mu I_\nu d\Omega = 4\pi H_\nu, \quad (8)$$

$$P_\nu = \frac{1}{c} \int \mu^2 I_\nu d\Omega = \frac{4\pi}{c} K_\nu. \quad (9)$$

The radiation energy density is

$$u_\nu = \frac{1}{c} \int I_\nu d\Omega = \frac{4\pi}{c} J_\nu. \quad (10)$$

*Remark 1.*  $J_\nu$  measures the angle-averaged radiative energy density,  $H_\nu$  measures net transport, and  $K_\nu$  measures the normal component of momentum flux. An isotropic radiation field has  $F_\nu = 0$  but nonzero pressure.

For an isotropic field  $I_\nu = I_{\nu,0}$ , direct integration gives

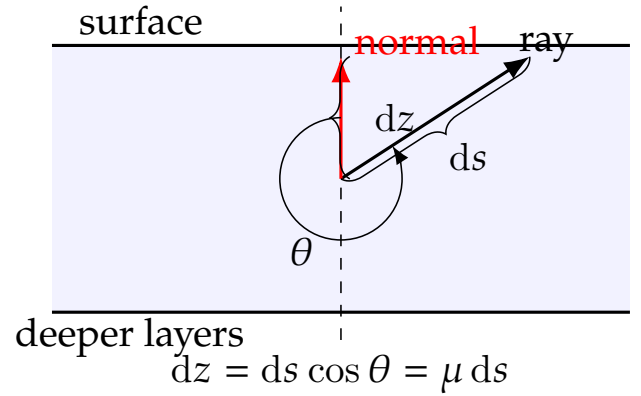
$$J_\nu = I_{\nu,0}, \quad (11)$$

$$H_\nu = 0, \quad (12)$$

$$K_\nu = \frac{I_{\nu,0}}{4\pi} \int \mu^2 d\Omega = \frac{I_{\nu,0}}{3}. \quad (13)$$

Thus

$$P_\nu = \frac{u_\nu}{3} \quad (14)$$



**Figure 1.** Ray geometry in a plane-parallel slab. The vertical coordinate  $z$  is normal to the layers, while  $s$  measures distance along the ray. The direction cosine is  $\mu = \cos \theta$ , so  $dz = \mu ds$ .

for isotropic radiation. The Eddington approximation extends this exact isotropic relation to mildly anisotropic radiation by setting  $K_\nu \approx J_\nu/3$ .

The blackbody intensity is the Planck function,

$$B_\nu(T) = \frac{2h\nu^3}{c^2} \left[ \exp\left(\frac{h\nu}{kT}\right) - 1 \right]^{-1}. \quad (15)$$

Its frequency integral is

$$B(T) = \int_0^\infty B_\nu d\nu = \frac{\sigma}{\pi} T^4, \quad (16)$$

where  $\sigma$  is the Stefan–Boltzmann constant. In thermodynamic equilibrium  $I_\nu = B_\nu$ ,  $J_\nu = B_\nu$ , and no net flux is present unless a temperature gradient breaks global equilibrium.

The radiation constant  $a$  is defined by integrating the isotropic blackbody energy density over frequency:

$$u_{\text{rad}} = \frac{4\pi}{c} B(T) = aT^4, \quad (17)$$

so that

$$a = \frac{4\sigma}{c}. \quad (18)$$

The corresponding equilibrium radiation pressure is

$$P_{\text{rad}} = \frac{1}{3} aT^4. \quad (19)$$

Although  $P_{\text{rad}}$  is small compared with gas pressure in the solar photosphere, it becomes dynamically important in massive stars, accretion disks, and radiation-dominated envelopes [6, 13].

## 3. Boltzmann Excitation from Statistical Weights

Consider atoms or ions of a single species in local thermodynamic equilibrium at temperature  $T$ . Let level  $i$  have excitation energy  $\epsilon_i$  measured from the ground state and degeneracy, or statistical weight,  $g_i$ . In the

canonical ensemble the probability of occupying a particular quantum state of energy  $\epsilon_i$  is proportional to  $\exp(-\epsilon_i/kT)$ . Since level  $i$  contains  $g_i$  states, the population of the level is proportional to

$$n_i \propto g_i \exp\left(-\frac{\epsilon_i}{kT}\right). \quad (20)$$

Normalization over all bound levels of the ionization stage gives the partition function

$$U(T) = \sum_i g_i \exp\left(-\frac{\epsilon_i}{kT}\right). \quad (21)$$

If  $n$  is the total number density in the ionization stage, then

$$n = \sum_i n_i. \quad (22)$$

Combining (20)–(22) gives

$$\frac{n_i}{n} = \frac{g_i}{U(T)} \exp\left(-\frac{\epsilon_i}{kT}\right). \quad (23)$$

For two levels  $i$  and  $j$  of the same ion,

$$\frac{n_j}{n_i} = \frac{g_j}{g_i} \exp\left[-\frac{\epsilon_j - \epsilon_i}{kT}\right]. \quad (24)$$

*Remark 2.* Equation (23) says that excitation is controlled by the competition between degeneracy and Boltzmann suppression. High levels may have large statistical weights, but if  $\epsilon_i \gg kT$  their populations remain small.

For hydrogenic bound levels,

$$g_n = 2n^2, \quad (25)$$

and the excitation energy relative to the ground state is

$$\epsilon_n = 13.598 \text{ eV} \left(1 - \frac{1}{n^2}\right). \quad (26)$$

Thus the  $n = 2$  level lies at

$$\epsilon_2 = 10.199 \text{ eV}. \quad (27)$$

Approximating the neutral-hydrogen partition function by the ground-level contribution,  $U_H \simeq g_1 = 2$ , yields

$$\frac{n_2}{n_{H^0}} \simeq \frac{8}{2} \exp\left(-\frac{10.199 \text{ eV}}{kT}\right). \quad (28)$$

At  $T = 9500 \text{ K}$ ,  $kT = 0.818 \text{ eV}$ , and therefore

$$\frac{n_2}{n_{H^0}} \simeq 4 \exp(-12.47) = 1.54 \times 10^{-5}. \quad (29)$$

Only a tiny fraction of neutral atoms occupy  $n = 2$ , but this fraction is much larger than in cooler stars.

A useful diagnostic of the validity of the Boltzmann formula is the comparison between collisional and radiative transition rates. For a lower level  $l$  and upper level  $u$ , LTE requires the upward and downward processes to be dominated by local collisions or by a Planckian radiation field. The collisional de-excitation rate per atom may be written

$$C_{ul} = n_e q_{ul}(T), \quad (30)$$

where  $q_{ul}$  is a collisional rate coefficient. A rough critical density follows by equating  $C_{ul}$  to the spontaneous rate  $A_{ul}$ :

$$n_{\text{crit}} = \frac{A_{ul}}{q_{ul}(T)}. \quad (31)$$

When  $n_e \gg n_{\text{crit}}$ , collisions strongly couple the levels to the local kinetic temperature. When  $n_e \ll n_{\text{crit}}$ , the level populations are controlled by the nonlocal radiation field and the Boltzmann ratio need not hold. This simple comparison explains why LTE is often adequate in deep photospheric layers but fails in chromospheres, winds, and nebulae.

#### 4. Saha Ionization from Continuum States

The Saha equation is often quoted as an independent result, but it is most transparently derived as a Boltzmann equation in which the upper “level” is the continuum of ion plus free electron states. Consider ionization stage  $r$  and the next stage  $r + 1$ :

$$X^r \rightleftharpoons X^{r+1} + e^-. \quad (32)$$

Let  $\chi_r$  be the ionization energy from the ground state of stage  $r$ . A free electron with momentum between  $\mathbf{p}$  and  $\mathbf{p} + d^3\mathbf{p}$  has kinetic energy

$$\epsilon_p = \frac{p^2}{2m_e}. \quad (33)$$

The number of one-particle translational states per unit volume in phase space is

$$\frac{d^3p \, d^3x}{h^3}. \quad (34)$$

Per unit volume, including the electron spin degeneracy 2, the density of electron states in  $d^3p$  is

$$\frac{2 \, d^3p}{h^3}. \quad (35)$$

The statistical weight of the combined ion-plus-electron continuum is the product of the ionic internal partition function  $U_{r+1}$  and the free-electron state count. Relative to the lower ionization stage with partition function  $U_r$ , the differential population ratio is therefore

$$\frac{dn_{r+1,e}}{n_r} = \frac{U_{r+1}}{U_r} \frac{2 \, d^3p}{h^3} \exp\left[-\frac{\chi_r + p^2/(2m_e)}{kT}\right]. \quad (36)$$

Integrating over all momenta gives

$$\frac{n_{r+1}n_e}{n_r} = \frac{2U_{r+1}}{U_r} \exp\left(-\frac{\chi_r}{kT}\right) \frac{1}{h^3} \int \exp\left(-\frac{p^2}{2m_e kT}\right) d^3p. \quad (37)$$

The three-dimensional Gaussian integral factorizes:

$$\int \exp\left(-\frac{p^2}{2m_e kT}\right) d^3p = \left[ \int_{-\infty}^{\infty} \exp\left(-\frac{p_x^2}{2m_e kT}\right) dp_x \right]^3$$

$$= (2\pi m_e kT)^{3/2}. \quad (38)$$

Substitution into (37) yields the Saha ionization equation:

$$\frac{n_{r+1}n_e}{n_r} = \frac{2U_{r+1}(T)}{U_r(T)} \left( \frac{2\pi m_e kT}{h^2} \right)^{3/2} \exp\left(-\frac{\chi_r}{kT}\right). \quad (39)$$

*Remark 3.* The factor  $2(2\pi m_e kT/h^2)^{3/2}$  is not arbitrary. It is the thermal density of available free-electron quantum states, including spin. The exponential penalizes ionization energy, while the  $T^{3/2}$  factor favours ionization because the continuum phase space grows with temperature.

The same expression may be written with the electron thermal de Broglie wavelength

$$\lambda_e = \frac{h}{(2\pi m_e kT)^{1/2}}, \quad (40)$$

which gives

$$\frac{n_{r+1}n_e}{n_r} = \frac{2U_{r+1}}{U_r} \lambda_e^{-3} \exp\left(-\frac{\chi_r}{kT}\right). \quad (41)$$

This form makes the quantum origin of the Saha factor especially clear. The quantity  $\lambda_e^3$  is the characteristic volume occupied by one thermal electron wave packet. Classical, nondegenerate statistics require

$$n_e \lambda_e^3 \ll 1. \quad (42)$$

In ordinary stellar photospheres this condition is well satisfied. It fails in dense white-dwarf interiors, where electron degeneracy modifies the chemical potential and the simple Saha equation must be replaced by an equation of state containing Fermi–Dirac statistics [7, 14].

For hydrogen, take  $U_p = 1$  and  $U_H \simeq 2$ . The ratio  $2U_p/U_H$  is then approximately unity, although it is common to keep the explicit electron-spin factor in the form

$$\Phi_H(T) = 2 \left( \frac{2\pi m_e kT}{h^2} \right)^{3/2} \frac{U_p}{U_H} \exp\left(-\frac{\chi_H}{kT}\right). \quad (43)$$

The ionization balance is

$$\frac{n_p n_e}{n_{H^0}} = \Phi_H(T), \quad (44)$$

with  $\chi_H = 13.598$  eV.

If hydrogen is the only electron donor and  $n_H = n_{H^0} + n_p$ , define the ionization fraction

$$y = \frac{n_p}{n_H}. \quad (45)$$

At fixed electron pressure,

$$P_e = n_e kT, \quad (46)$$

one has

$$\frac{y}{1-y} = \frac{n_p}{n_{H^0}} = \frac{\Phi_H(T)}{n_e} = \frac{kT \Phi_H(T)}{P_e}. \quad (47)$$

Hence

$$y(T; P_e) = \frac{A(T; P_e)}{1 + A(T; P_e)}, \quad (48)$$

where

$$A(T; P_e) = \frac{kT}{P_e} 2 \left( \frac{2\pi m_e kT}{h^2} \right)^{3/2} \frac{U_p}{U_H} \exp\left(-\frac{\chi_H}{kT}\right). \quad (49)$$

The lower-level population relevant for Balmer absorption is approximately

$$\frac{n_2}{n_H} = (1-y) \frac{g_2}{U_H} \exp\left(-\frac{10.199 \text{ eV}}{kT}\right). \quad (50)$$

This expression makes the Balmer maximum intuitive: the Boltzmann factor increases with  $T$ , while the neutral fraction  $1-y$  decreases with  $T$ .

## 5. Equation of Radiative Transfer

Let  $\eta_v$  be the emission coefficient per unit volume, frequency, time, and solid angle, and let  $\chi_v$  be the extinction coefficient per unit length. Along a ray coordinate  $s$ , the change in intensity over  $ds$  is

$$dI_v = -\chi_v I_v ds + \eta_v ds. \quad (51)$$

Dividing by  $ds$  gives the transfer equation

$$\frac{dI_v}{ds} = -\chi_v I_v + \eta_v. \quad (52)$$

Define the source function

$$S_v = \frac{\eta_v}{\chi_v}. \quad (53)$$

Then

$$\frac{dI_v}{ds} = -\chi_v (I_v - S_v). \quad (54)$$

The optical-depth increment along the ray is

$$d\tau_v = \chi_v ds. \quad (55)$$

If  $\tau_v$  increases opposite to the direction of photon propagation, the outward transfer equation is commonly written

$$\mu \frac{dI_v}{d\tau_v} = I_v - S_v, \quad (56)$$

where  $\tau_v$  is the vertical optical depth increasing inward.

The formal solution for outward radiation at the surface is found by multiplying (56) by the integrating factor  $\exp(-\tau_v/\mu)$ . For  $\mu > 0$ ,

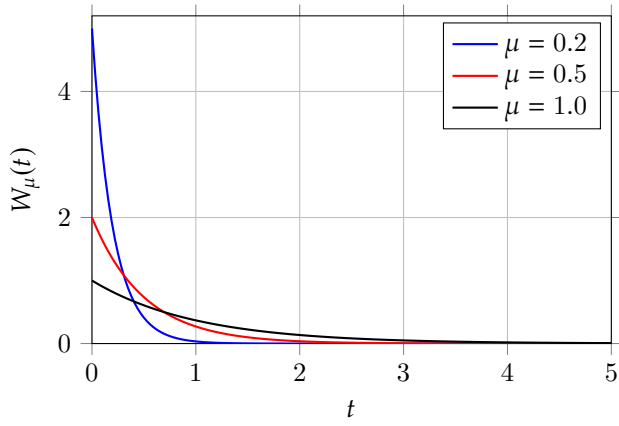
$$\frac{d}{d\tau_v} [I_v(\tau_v, \mu) e^{-\tau_v/\mu}] = -\frac{S_v(\tau_v)}{\mu} e^{-\tau_v/\mu}. \quad (57)$$

Integrating from  $\tau_v = 0$  to  $\infty$  and assuming the deep contribution is extinguished gives

$$I_v(0, \mu) = \frac{1}{\mu} \int_0^\infty S_v(t) e^{-t/\mu} dt. \quad (58)$$

More generally, between optical depths  $\tau_1$  and  $\tau_2$ ,

$$I_v(\tau_1, \mu) = I_v(\tau_2, \mu) e^{-(\tau_2 - \tau_1)/\mu} + \frac{1}{\mu} \int_{\tau_1}^{\tau_2} S_v(t) e^{-(t - \tau_1)/\mu} dt. \quad (59)$$



**Figure 2.** Contribution kernels  $W_\mu(t) = \mu^{-1} \exp(-t/\mu)$  for emergent rays. Limb rays with small  $\mu$  receive most of their intensity from very shallow optical depths, whereas disk-centre rays sample deeper layers.

*Remark 4.* Equation (58) says that the emergent intensity at angle  $\mu$  samples the source function over an optical-depth range  $t \sim \mu$ . Near the limb, where  $\mu$  is small, the observer sees shallower and cooler layers; this is the origin of limb darkening.

The exponential in (58) defines a normalized contribution kernel

$$W_\mu(t) = \frac{1}{\mu} e^{-t/\mu}, \quad (60)$$

because

$$\int_0^\infty W_\mu(t) dt = 1. \quad (61)$$

The first moment of this kernel is

$$\int_0^\infty t W_\mu(t) dt = \mu. \quad (62)$$

Thus the optical depth most strongly sampled by a ray is of order  $\mu$ , not exactly unity for all directions. If the source function is locally linear,

$$S_\nu(t) = a_\nu + b_\nu t, \quad (63)$$

then the formal solution gives

$$\begin{aligned} I_\nu(0, \mu) &= a_\nu + b_\nu \mu \\ &= S_\nu(\tau_\nu = \mu). \end{aligned} \quad (64)$$

Equation (64) is the Eddington–Barbier relation. It is not a separate approximation when  $S_\nu$  is exactly linear; rather, it is the exact evaluation of the transfer integral for that source function. In real atmospheres it is an excellent local rule of thumb for continuum radiation.

In local thermodynamic equilibrium, Kirchhoff's law gives

$$S_\nu = B_\nu(T) \quad (65)$$

for true absorption and thermal emission. With coherent isotropic scattering,

$$S_\nu = (1 - \epsilon_\nu) J_\nu + \epsilon_\nu B_\nu, \quad (66)$$

where  $\epsilon_\nu$  is the photon destruction probability. The grey atmosphere derived below assumes the frequency-integrated version of (65) in radiative equilibrium.

## 6. Plane-Parallel Moment Equations

In plane-parallel geometry all mean quantities depend only on vertical optical depth. Let  $z$  increase outward and let  $\tau_\nu$  increase inward:

$$d\tau_\nu = -\chi_\nu dz. \quad (67)$$

Using  $dz = \mu ds$  gives the transfer equation

$$\mu \frac{\partial I_\nu}{\partial \tau_\nu} = I_\nu - S_\nu. \quad (68)$$

Take the zeroth angular moment by integrating (68) over solid angle and dividing by  $4\pi$ :

$$\frac{1}{4\pi} \int \mu \frac{\partial I_\nu}{\partial \tau_\nu} d\Omega = \frac{1}{4\pi} \int (I_\nu - S_\nu) d\Omega. \quad (69)$$

If  $S_\nu$  is isotropic, then

$$\frac{\partial H_\nu}{\partial \tau_\nu} = J_\nu - S_\nu. \quad (70)$$

Now multiply (68) by  $\mu$ , integrate, and divide by  $4\pi$ :

$$\frac{1}{4\pi} \int \mu^2 \frac{\partial I_\nu}{\partial \tau_\nu} d\Omega = \frac{1}{4\pi} \int \mu (I_\nu - S_\nu) d\Omega. \quad (71)$$

The source term vanishes because  $\int \mu d\Omega = 0$ , giving

$$\frac{\partial K_\nu}{\partial \tau_\nu} = H_\nu. \quad (72)$$

For frequency-integrated grey transfer, define

$$J = \int_0^\infty J_\nu d\nu, \quad (73)$$

$$H = \int_0^\infty H_\nu d\nu, \quad (74)$$

$$K = \int_0^\infty K_\nu d\nu. \quad (75)$$

The grey approximation assumes that the opacity used to define optical depth is independent of frequency:

$$\chi_\nu \rightarrow \chi. \quad (76)$$

The moment equations become

$$\frac{dH}{d\tau} = J - S, \quad (77)$$

$$\frac{dK}{d\tau} = H. \quad (78)$$

Radiative equilibrium requires that matter neither gains nor loses net energy by radiation:

$$\int_0^\infty \chi_\nu (J_\nu - S_\nu) d\nu = 0. \quad (79)$$

In the grey case this reduces to

$$J = S. \quad (80)$$

Therefore (77) gives

$$\frac{dH}{d\tau} = 0, \quad (81)$$

so the flux is constant with depth. The total emergent flux defines the effective temperature:

$$F = 4\pi H = \sigma T_{\text{eff}}^4. \quad (82)$$

Thus

$$H = \frac{\sigma T_{\text{eff}}^4}{4\pi}. \quad (83)$$

The same moment equations yield the diffusion approximation in the optically thick limit. When the intensity is nearly isotropic, write the leading angular dependence as

$$I(\tau, \mu) = J(\tau) + 3\mu H(\tau). \quad (84)$$

This form reproduces the moments  $J$  and  $H$  to first order in the anisotropy. Substituting a slowly varying Planck source into the transfer equation gives

$$H \simeq \frac{1}{3} \frac{dB}{d\tau}. \quad (85)$$

Because  $d\tau = -\kappa\rho dz$  and  $B = \sigma T^4/\pi$ , the physical flux becomes

$$F = 4\pi H = -\frac{16\sigma T^3}{3\kappa\rho} \frac{dT}{dz}. \quad (86)$$

Using  $a = 4\sigma/c$ , this is equivalently

$$F = -\frac{4acT^3}{3\kappa\rho} \frac{dT}{dz}. \quad (87)$$

The negative sign indicates that radiative energy flows down the temperature gradient.

## 7. Eddington Approximation and Grey Temperature Law

The moment hierarchy is not closed because (78) contains both  $K$  and  $H$ . The Eddington approximation closes it by assuming

$$K = \frac{1}{3}J. \quad (88)$$

Since radiative equilibrium gives  $J = S$ , and LTE thermal emission gives

$$S = B = \frac{\sigma}{\pi} T^4, \quad (89)$$

we obtain

$$K = \frac{1}{3} \frac{\sigma}{\pi} T^4. \quad (90)$$

From (78),

$$\frac{dK}{d\tau} = H = \frac{\sigma T_{\text{eff}}^4}{4\pi}. \quad (91)$$

Integrating gives

$$K(\tau) = H\tau + C. \quad (92)$$

The integration constant is set by the Eddington boundary condition. At the surface, no incoming radiation is assumed:

$$I(0, \mu) = 0 \quad \text{for } \mu < 0. \quad (93)$$

The outgoing radiation is approximated as weakly anisotropic in such a way that

$$J(0) = 2H(0). \quad (94)$$

Using  $K(0) = J(0)/3$  gives

$$K(0) = \frac{2}{3}H. \quad (95)$$

Therefore

$$K(\tau) = H \left( \tau + \frac{2}{3} \right). \quad (96)$$

Combining (90), (83), and (96),

$$\frac{1}{3} \frac{\sigma}{\pi} T^4 = \frac{\sigma T_{\text{eff}}^4}{4\pi} \left( \tau + \frac{2}{3} \right), \quad (97)$$

$$T^4(\tau) = \frac{3}{4} T_{\text{eff}}^4 \left( \tau + \frac{2}{3} \right). \quad (98)$$

Equivalently,

$$\frac{T(\tau)}{T_{\text{eff}}} = \left[ \frac{3}{4} \left( \tau + \frac{2}{3} \right) \right]^{1/4}. \quad (99)$$

**Proposition 1** (Grey photosphere). *In the Eddington grey atmosphere, the layer with local temperature equal to the effective temperature lies at optical depth*

$$\tau_{\text{ph}} = \frac{2}{3}. \quad (100)$$

*Proof.* Set  $T(\tau) = T_{\text{eff}}$  in (98). Then

$$1 = \frac{3}{4} \left( \tau + \frac{2}{3} \right), \quad (101)$$

so

$$\tau + \frac{2}{3} = \frac{4}{3}, \quad (102)$$

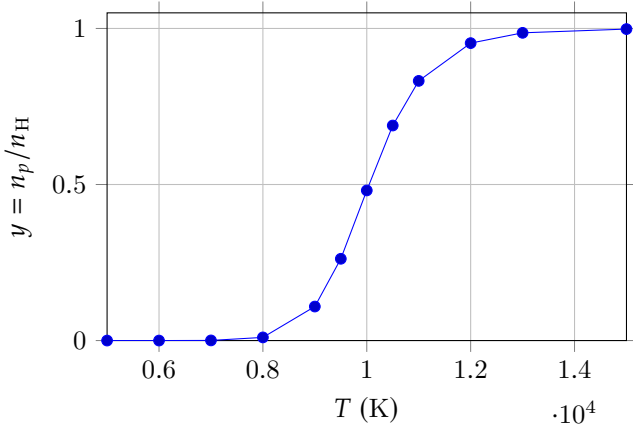
and hence  $\tau = 2/3$ .  $\square$

The photosphere is therefore not the  $\tau = 0$  surface. It is the depth from which photons representative of the emergent flux escape. In a non-grey atmosphere the optical depth is frequency-dependent, so different wavelengths have different photospheres.

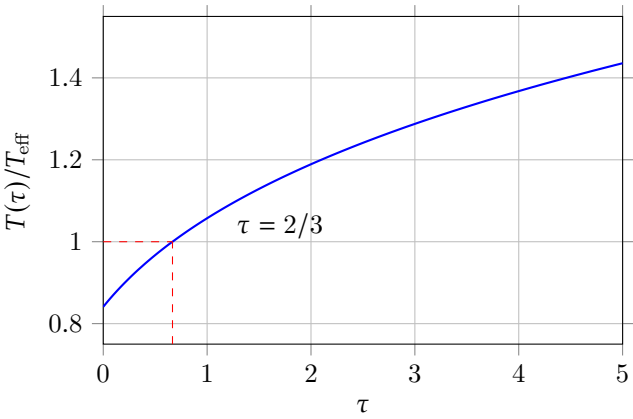
The Eddington factor is often written as

$$f(\tau) = \frac{K(\tau)}{J(\tau)}. \quad (103)$$

The closure (88) is the assumption  $f = 1/3$ . At the top of a purely outward beam, however,  $K/J$  can approach unity, whereas for an exactly isotropic field it is exactly  $1/3$ . The grey solution therefore combines a deep, diffusion-like angular closure with a surface boundary condition intended to mimic the one-sided escape of photons. This hybrid nature explains both its success as a scaling law and its limited quantitative accuracy.



**Figure 3.** Hydrogen ionization fraction from the Saha equation at fixed electron pressure  $P_e = 10^3 \text{ dyn cm}^{-2}$ , with  $U_H \simeq 2$  and  $U_p = 1$ . The rapid transition reflects the exponential sensitivity to  $\chi_H/kT$ .



**Figure 4.** Temperature structure of the Eddington grey atmosphere. The local temperature equals  $T_{\text{eff}}$  at  $\tau = 2/3$ , motivating the classical photospheric optical depth.

## 8. Emergent Intensity and Limb Darkening

The formal solution (58) may be evaluated analytically in the grey atmosphere. Since  $S = B = J$  and  $J = 3K$ , equations (96) and (80) imply

$$S(\tau) = J(\tau) = 3H \left( \tau + \frac{2}{3} \right). \quad (104)$$

Substitution into (58) gives

$$I(0, \mu) = \frac{1}{\mu} \int_0^\infty 3H \left( t + \frac{2}{3} \right) e^{-t/\mu} dt. \quad (105)$$

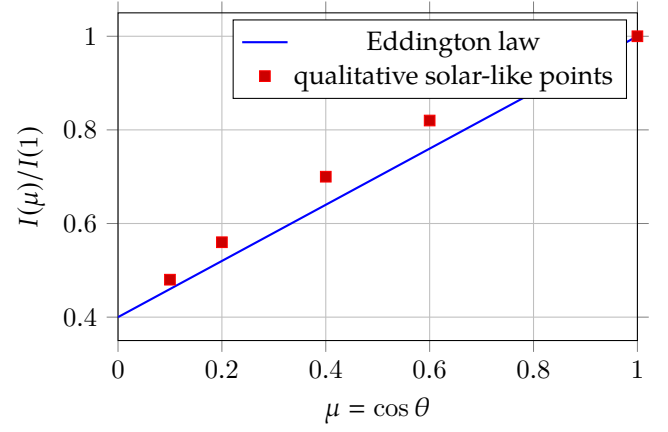
The required integrals are

$$\int_0^\infty e^{-t/\mu} dt = \mu, \quad (106)$$

$$\int_0^\infty t e^{-t/\mu} dt = \mu^2. \quad (107)$$

Therefore

$$I(0, \mu) = \frac{3H}{\mu} \left[ \mu^2 + \frac{2}{3}\mu \right]$$



**Figure 5.** Eddington limb-darkening law  $I(\mu)/I(1) = (2 + 3\mu)/5$ . The red points are illustrative solar-like values showing that real wavelength-dependent limb darkening is similar in trend but not exactly described by the grey linear law.

$$= 3H \left( \mu + \frac{2}{3} \right). \quad (108)$$

At disk centre  $\mu = 1$ ,

$$I(0, 1) = 3H \left( \frac{5}{3} \right) = 5H. \quad (109)$$

Thus the Eddington limb-darkening law is

$$\frac{I(\mu)}{I(1)} = \frac{\mu + 2/3}{1 + 2/3} = \frac{2 + 3\mu}{5}. \quad (110)$$

*Remark 5.* Equation (110) is linear in  $\mu$ . It predicts  $I(0)/I(1) = 0.4$ , whereas solar limb darkening depends strongly on wavelength and is not exactly linear. The qualitative trend is correct: rays near the limb escape from smaller optical depths and hence cooler layers.

The observable flux is recovered by integrating the emergent intensity over the outward hemisphere:

$$F = 2\pi \int_0^1 I(0, \mu) \mu d\mu. \quad (111)$$

Using (108),

$$\begin{aligned} F &= 6\pi H \int_0^1 \left( \mu^2 + \frac{2}{3}\mu \right) d\mu \\ &= 6\pi H \left( \frac{1}{3} + \frac{1}{3} \right) = 4\pi H. \end{aligned} \quad (112)$$

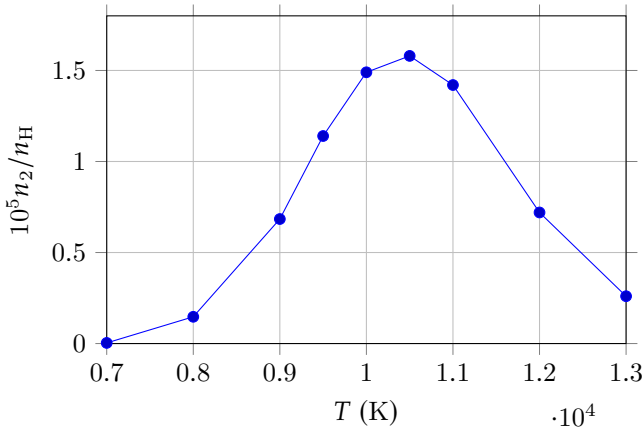
The limb-darkening law is therefore consistent with the flux normalization used in defining  $T_{\text{eff}}$ . This check is important: an empirical limb-darkening fit used in stellar-radius or exoplanet-transit work must conserve the total flux if it is to be interpreted as a surface-brightness law.

## 9. Worked Numerical Applications

### 9.1. Hydrogen ionization at fixed electron pressure

Use cgs units. Let

$$m_e = 9.109 \times 10^{-28} \text{ g}, \quad (113)$$



**Figure 6.** Approximate Balmer lower-level population from (50), scaled by  $10^5$ , for the same fixed electron pressure as Fig. 3. The maximum near  $10^4$  K reflects the competition between excitation and ionization and explains the strong Balmer lines of A-type stars.

$$k = 1.381 \times 10^{-16} \text{ erg K}^{-1}, \quad (114)$$

$$h = 6.626 \times 10^{-27} \text{ erg s}, \quad (115)$$

$$\chi_H = 13.598 \text{ eV} = 2.179 \times 10^{-11} \text{ erg}. \quad (116)$$

The electron-state factor is

$$2 \left( \frac{2\pi m_e k}{h^2} \right)^{3/2} = 4.83 \times 10^{15} \text{ cm}^{-3} \text{ K}^{-3/2}. \quad (117)$$

With  $U_p/U_H = 1/2$ , the effective hydrogen factor is approximately

$$\Phi_H(T) = 2.42 \times 10^{15} T^{3/2} \exp \left( -\frac{1.5789 \times 10^5}{T} \right) \text{ cm}^{-3}. \quad (118)$$

If instead the explicit expression (43) is retained, the same convention must be used consistently for  $U_H$ . At  $P_e = 10^3 \text{ dyn cm}^{-2}$ ,

$$n_e = \frac{P_e}{kT} = 7.24 \times 10^{14} \left( \frac{10^4 \text{ K}}{T} \right) \text{ cm}^{-3}. \quad (119)$$

At  $T = 10^4$  K, using the convention of Fig. 3,

$$A = \frac{kT\Phi_H}{P_e} \simeq 0.93, \quad (120)$$

$$y = \frac{0.93}{1 + 0.93} \simeq 0.48. \quad (121)$$

Thus hydrogen is roughly half ionized. At  $T = 9000$  K,

$$y \simeq 0.11, \quad (122)$$

whereas at  $T = 11000$  K,

$$y \simeq 0.83. \quad (123)$$

The transition is narrow because of the exponential term  $\exp(-\chi/kT)$ .

## 9.2. Balmer-line strength maximum

For the  $n = 2$  population, combine Saha ionization with Boltzmann excitation:

$$\frac{n_2}{n_H} = (1 - y) 4 \exp \left( -\frac{1.1837 \times 10^5}{T} \right). \quad (124)$$

At  $T = 9500$  K, Fig. 3 gives  $y \simeq 0.262$ , so

$$\begin{aligned} \frac{n_2}{n_H} &= 0.738 \times 4 \exp(-12.46) \\ &= 1.14 \times 10^{-5}. \end{aligned} \quad (125)$$

At  $T = 10500$  K,

$$\begin{aligned} \frac{n_2}{n_H} &= 0.311 \times 4 \exp(-11.27) \\ &= 1.58 \times 10^{-5}. \end{aligned} \quad (126)$$

At still higher temperature, ionization wins. For example, at  $T = 12000$  K,

$$\frac{n_2}{n_H} \simeq 7.2 \times 10^{-6}. \quad (127)$$

The maximum in this simplified fixed-pressure calculation is near  $10^4$  K; detailed stellar atmospheres place the observed Balmer maximum near A0 spectral type, about 9500 K, because pressure, opacity, line broadening, and depth-dependent formation alter the simple estimate.

## 9.3. Grey atmosphere temperatures

For the Sun, take

$$T_{\text{eff},\odot} = 5772 \text{ K}. \quad (128)$$

At the surface  $\tau = 0$ ,

$$\begin{aligned} T(0) &= T_{\text{eff}} \left( \frac{1}{2} \right)^{1/4} \\ &= 0.841 T_{\text{eff}} \simeq 4850 \text{ K}. \end{aligned} \quad (129)$$

At  $\tau = 2/3$ ,

$$T = T_{\text{eff}} = 5772 \text{ K}. \quad (130)$$

At  $\tau = 10$ ,

$$\begin{aligned} T(10) &= 5772 \left[ \frac{3}{4} \left( 10 + \frac{2}{3} \right) \right]^{1/4} \text{ K} \\ &= 9.69 \times 10^3 \text{ K}. \end{aligned} \quad (131)$$

The grey model therefore predicts a steep rise below the photosphere, but it should not be extrapolated indefinitely because convection, non-grey opacity, and diffusion physics become important.

## 9.4. A solar photospheric pressure scale

The optical-depth coordinate may be connected to gas pressure by hydrostatic equilibrium. In a thin plane-parallel layer,

$$\frac{dP}{dz} = -\rho g, \quad (132)$$

and the grey optical depth obeys

$$d\tau = -\kappa\rho dz, \quad (133)$$

where  $\kappa$  is the mass extinction coefficient. Combining the two equations gives

$$\frac{dP}{d\tau} = \frac{g}{\kappa}. \quad (134)$$

If  $g$  and  $\kappa$  are approximately constant over the photospheric layer and  $P(0)$  is neglected, then

$$P(\tau) \simeq \frac{g\tau}{\kappa}. \quad (135)$$

For a solar example, take

$$g_{\odot} = 2.74 \times 10^4 \text{ cm s}^{-2}, \quad (136)$$

$$\kappa = 0.20 \text{ cm}^2 \text{ g}^{-1}. \quad (137)$$

At the grey photosphere  $\tau = 2/3$ ,

$$P_{\text{ph}} \simeq 9.1 \times 10^4 \text{ dyn cm}^{-2}. \quad (138)$$

Assuming an ideal gas with mean molecular weight  $\mu_m = 1.3$ ,

$$\rho = \frac{\mu_m m_{\text{H}} P}{kT}. \quad (139)$$

With  $T = 5772 \text{ K}$ , this gives

$$\rho_{\text{ph}} \simeq 2.5 \times 10^{-7} \text{ g cm}^{-3}. \quad (140)$$

The corresponding photon mean free path is

$$\ell = \frac{1}{\kappa\rho_{\text{ph}}} \simeq 2.0 \times 10^7 \text{ cm}, \quad (141)$$

or about 200 km. This number is comparable with the vertical scale over which photospheric temperature and density change, illustrating why the stellar surface is extended rather than sharp.

The gas pressure scale height is

$$H_P = \frac{kT}{\mu_m m_{\text{H}} g_{\odot}} \simeq 1.3 \times 10^7 \text{ cm}. \quad (142)$$

The comparison  $\ell \sim H_P$  at optical depth of order unity gives a useful physical interpretation of the photosphere: a photon travels a macroscopic atmospheric scale before being absorbed or scattered.

## 10. Extensions and Observational Tests

The grey atmosphere is best regarded as the first term in a sequence of increasingly realistic models. The most immediate generalization is monochromatic transfer. At each frequency one defines

$$d\tau_{\nu} = -\kappa_{\nu}\rho dz, \quad (143)$$

so that the approximate continuum formation depth satisfies

$$\tau_{\nu} \simeq \frac{2}{3}. \quad (144)$$

**Table 1.** Representative numerical values for hydrogen at  $P_e = 10^3 \text{ dyn cm}^{-2}$ .

| $T \text{ (K)}$ | $y = n_p/n_{\text{H}}$ | $10^5 n_2/n_{\text{H}}$ |
|-----------------|------------------------|-------------------------|
| 8000            | 0.010                  | 0.147                   |
| 9000            | 0.109                  | 0.684                   |
| 9500            | 0.262                  | 1.14                    |
| 10000           | 0.481                  | 1.49                    |
| 10500           | 0.689                  | 1.58                    |
| 11000           | 0.832                  | 1.42                    |
| 12000           | 0.953                  | 0.72                    |

Because  $\kappa_{\nu}$  varies strongly with wavelength, the condition (144) selects different geometrical heights at different frequencies. A bound-free edge, for instance, can move the ultraviolet photosphere upward into cooler material, producing an abrupt change in emergent flux.

For an unresolved spherical star at distance  $D$ , the observed monochromatic flux is related to the surface flux by

$$f_{\nu} = \left(\frac{R}{D}\right)^2 F_{\nu}, \quad (145)$$

where  $R$  is the stellar radius. If the surface intensity is known as a function of  $\mu$ , then

$$F_{\nu} = 2\pi \int_0^1 I_{\nu}(0, \mu) \mu d\mu. \quad (146)$$

Equations (145) and (146) are the basis for angular-diameter measurements, spectrophotometric calibration, and stellar radius estimates when combined with parallax.

A useful observational construct is the brightness temperature  $T_b(\nu)$ , defined implicitly by

$$I_{\nu} = B_{\nu}(T_b). \quad (147)$$

Solving the Planck function for  $T_b$  gives

$$T_b(\nu) = \frac{h\nu}{k} \left[ \ln \left( 1 + \frac{2h\nu^3}{c^2 I_{\nu}} \right) \right]^{-1}. \quad (148)$$

In LTE continuum regions the Eddington–Barbier relation implies that  $T_b(\nu, \mu)$  approximately measures the gas temperature at  $\tau_{\nu} \simeq \mu$ . Center-to-limb observations of the Sun therefore provide a direct empirical test of the vertical temperature gradient in a model atmosphere.

Spectral lines add another layer of diagnostics. The total extinction near a line may be written

$$\kappa_{\nu} = \kappa_c + \kappa_l \phi(\nu), \quad (149)$$

where  $\kappa_c$  is the continuum opacity,  $\kappa_l$  is the integrated line opacity, and  $\phi(\nu)$  is a normalized profile:

$$\int_0^{\infty} \phi(\nu) d\nu = 1. \quad (150)$$

The equivalent width of an absorption line is

$$W_\lambda = \int \left(1 - \frac{F_\lambda}{F_{\lambda,c}}\right) d\lambda. \quad (151)$$

In the weak-line limit the equivalent width is proportional to the column density  $N_l$  of absorbers in the lower level:

$$W_\lambda \propto f_{lu} N_l, \quad (152)$$

where  $f_{lu}$  is the oscillator strength. The lower-level column density depends on abundance, ionization fraction, and Boltzmann excitation. This is why abundance analysis requires the Saha-Boltzmann factors derived above in addition to radiative transfer.

Limb darkening is now tested very accurately by eclipsing binaries, optical interferometry, microlensing caustic crossings, and exoplanet transits. A common empirical law is the linear form

$$\frac{I(\mu)}{I(1)} = 1 - u_\lambda(1 - \mu), \quad (153)$$

where  $u_\lambda$  is wavelength dependent. The Eddington law (110) corresponds to

$$u_\lambda = \frac{3}{5} \quad (154)$$

in this notation. Real values vary with wavelength, temperature, gravity, and metallicity; their departures from (154) are direct evidence of non-grey opacity and non-linear source functions.

Atmospheric extension becomes important when the pressure scale height is not negligible compared with the stellar radius. In spherical symmetry the transfer equation contains an angular-advection term:

$$\mu \frac{\partial I_\nu}{\partial r} + \frac{1 - \mu^2}{r} \frac{\partial I_\nu}{\partial \mu} = -\chi_\nu(I_\nu - S_\nu). \quad (155)$$

The plane-parallel equation is recovered when the local scale height is much smaller than  $r$ , so that curvature terms are negligible. Spherical transfer is required in extended giants, supergiants, novae, and expanding winds.

Finally, non-LTE modeling replaces Saha-Boltzmann populations by statistical-equilibrium equations. For level  $i$ ,

$$\sum_{j \neq i} n_j P_{ji} = n_i \sum_{j \neq i} P_{ij}, \quad (156)$$

where  $P_{ij}$  includes radiative and collisional rates. These equations are coupled to the transfer equation because radiative rates depend on the radiation field, while the opacity and emissivity depend on the level populations. The classical LTE equations are recovered when detailed balance forces all rates to produce the Boltzmann and Saha ratios at the local temperature.

## 11. Discussion

The derivations above rest on a hierarchy of approximations. The Boltzmann and Saha equations require thermodynamic equilibrium among material particles, or at least local thermodynamic equilibrium. Collisions must be frequent enough to establish Maxwellian electron velocities and level populations characterized by a local temperature. In the tenuous upper atmospheres of hot stars, planetary nebulae, chromospheres, and supernova ejecta, radiative rates can dominate collisional rates. Then the level populations must be found from statistical-equilibrium equations rather than from (23) and (39) [8, 15].

The partition function also deserves care. A naive hydrogenic sum over bound states diverges because  $g_n = 2n^2$  grows without limit while the level energies approach the continuum. In real plasmas, high levels are dissolved by electric microfields, pressure ionization, and finite-density effects. Practical equations of state therefore use occupation probabilities or explicit truncations. For the simple Balmer example the approximation  $U_H \simeq 2$  is adequate because the ground state dominates the neutral partition function at the relevant temperatures.

The grey atmosphere assumes frequency-independent opacity. Real stellar opacities are strongly non-grey: bound-free edges, free-free absorption,  $H^-$  opacity, molecular bands, and millions of spectral lines produce wavelength-dependent formation depths. The Sun's visible continuum is shaped largely by  $H^-$  opacity, while ultraviolet radiation forms higher in cooler layers because metal line blanketing and bound-free opacity are large. Consequently the relation between colour temperature, brightness temperature, and effective temperature is not captured by a single grey optical depth.

The Eddington closure  $K = J/3$  is exact only for an isotropic radiation field. Deep in an optically thick atmosphere this is justified by diffusion: the radiation field is nearly isotropic, with a small anisotropic correction carrying the flux. Near the surface the field becomes strongly one-sided, and the closure becomes approximate. The boundary condition  $J(0) = 2H(0)$  is likewise approximate. More accurate grey solutions introduce the Hopf function  $q(\tau)$ :

$$T^4(\tau) = \frac{3}{4} T_{\text{eff}}^4 [\tau + q(\tau)]. \quad (157)$$

For the Eddington solution,

$$q(\tau) = \frac{2}{3}. \quad (158)$$

Exact grey radiative-equilibrium solutions give a depth-dependent  $q(\tau)$ , with representative values ranging from about

$$q(0) \simeq 0.577 \quad (159)$$

to

$$q(\infty) \simeq 0.710. \quad (160)$$

The Eddington value 0.667 is therefore a useful average rather than a theorem of transfer theory.

Observed solar limb darkening illustrates the same point. The Eddington law (110) predicts a simple linear dependence on  $\mu$  and a wavelength-independent limb intensity of 0.4 times the disk-centre intensity. Actual solar limb darkening is stronger in the blue and weaker in the infrared; it is better represented by polynomial, logarithmic, square-root, or four-parameter laws. The physical reason remains the one captured by the grey derivation: oblique rays escape from smaller optical depths, and temperature decreases outward. The quantitative law changes because the source function is not exactly linear in optical depth and because monochromatic opacity selects different heights.

Despite these limitations, the analytic grey atmosphere remains indispensable. It explains why  $T_{\text{eff}}$  is associated with an optical depth of order unity rather than with the outer boundary. It provides the outer boundary condition for many stellar-structure calculations [13]. It also clarifies the diffusion approximation used deep inside stars. In the optically thick limit, combining (9), (14), and the first-order anisotropic correction to the transfer equation gives the radiative flux down the temperature gradient:

$$F = -\frac{4acT^3}{3\kappa\rho} \frac{dT}{dz}, \quad (161)$$

where  $a = 4\sigma/c$  is the radiation constant and  $\kappa\rho = \chi$ . This expression is the stellar-structure form of radiative energy transport.

Modern stellar-atmosphere models generalize every element of the classical theory. They solve transfer at many frequencies, include scattering, line blanketing, convection, spherical extension, winds, magnetic fields, and non-LTE rate equations. Nevertheless, their numerical output is interpreted with the same concepts derived here: intensity moments, source functions, optical-depth surfaces, excitation and ionization equilibria, and the angular structure of emergent radiation.

## 12. Conclusion

We have derived the Saha-Boltzmann equations and the Eddington grey atmosphere from their statistical and radiative foundations. The Boltzmann formula follows from canonical weighting of bound levels by statistical weight and excitation energy. The Saha equation follows by treating ionization as excitation into the continuum and integrating the free-electron phase-space density, producing the characteristic factor  $2(2\pi m_e kT/h^2)^{3/2}$ . Applied to hydrogen at fixed electron pressure, these equations show that Balmer absorption is maximized near  $10^4$  K because excitation into  $n = 2$  rises with temperature while the neutral fraction falls.

The transfer equation expresses the local competition between extinction and emission. Its formal solution

shows that emergent radiation samples optical depths of order  $\mu$ , leading directly to the Eddington-Barbier interpretation of continuum formation. Taking angular moments in a plane-parallel grey atmosphere and imposing radiative equilibrium gives constant flux. With the Eddington closure and boundary condition one obtains

$$T^4 = \frac{3}{4} T_{\text{eff}}^4 (\tau + 2/3), \quad (162)$$

the photospheric condition  $\tau = 2/3$ , and the limb-darkening law  $I(\mu)/I(1) = (2 + 3\mu)/5$ . These results are approximate but physically transparent. Non-grey opacity, scattering, non-LTE populations, spherical extension, and realistic surface boundary conditions modify their numerical predictions, yet the analytic structure remains the foundation of stellar-atmosphere theory.

## A. Detailed Algebra for Selected Results

This appendix collects algebraic steps that are often suppressed in compact derivations.

First, the Gaussian integral used in the Saha equation follows from the square of a one-dimensional integral. Let

$$G(a) = \int_{-\infty}^{\infty} e^{-ax^2} dx, \quad a > 0. \quad (163)$$

Then

$$\begin{aligned} G^2(a) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-a(x^2+y^2)} dx dy \\ &= 2\pi \int_0^{\infty} r e^{-ar^2} dr = \frac{\pi}{a}. \end{aligned} \quad (164)$$

Thus

$$G(a) = \left(\frac{\pi}{a}\right)^{1/2}. \quad (165)$$

With

$$a = \frac{1}{2m_e kT}, \quad (166)$$

the three-dimensional integral is

$$G^3(a) = (2\pi m_e kT)^{3/2}. \quad (167)$$

Second, the Eddington surface boundary condition can be motivated by a simple hemispheric approximation. Suppose the emergent intensity at the surface is constant over the outward hemisphere and zero over the inward hemisphere:

$$I(0, \mu) = \begin{cases} I_0, & 0 < \mu \leq 1, \\ 0, & -1 \leq \mu < 0. \end{cases} \quad (168)$$

Then

$$J(0) = \frac{1}{2} \int_0^1 I_0 d\mu = \frac{I_0}{2}, \quad (169)$$

$$H(0) = \frac{1}{2} \int_0^1 \mu I_0 d\mu = \frac{I_0}{4}. \quad (170)$$

Therefore

$$J(0) = 2H(0). \quad (171)$$

The assumption of constant outward intensity is crude, but the resulting relation captures the fact that the surface field is approximately hemispheric rather than isotropic.

Third, the flux conservation of the grey limb-darkening law follows directly from its normalization. With

$$I(\mu) = I(1) \frac{2 + 3\mu}{5}, \quad (172)$$

the flux is

$$\begin{aligned} F &= 2\pi I(1) \int_0^1 \frac{2 + 3\mu}{5} \mu \, d\mu \\ &= \frac{2\pi I(1)}{5} \left[ \int_0^1 2\mu \, d\mu + \int_0^1 3\mu^2 \, d\mu \right] \\ &= \frac{2\pi I(1)}{5} (1 + 1) = \frac{4\pi I(1)}{5}. \end{aligned} \quad (173)$$

Since (109) gives  $I(1) = 5H$ , this becomes

$$F = 4\pi H, \quad (174)$$

as required by the definition of  $H$ .

Finally, the grey temperature law may be written in column-mass form. Define the column mass above height  $z$  by

$$m(z) = \int_z^\infty \rho(z') \, dz'. \quad (175)$$

For constant grey opacity,

$$\tau = \kappa m. \quad (176)$$

Substitution into (98) gives

$$T^4(m) = \frac{3}{4} T_{\text{eff}}^4 \left( \kappa m + \frac{2}{3} \right). \quad (177)$$

This form is useful in stellar-envelope integrations, where hydrostatic balance with constant gravity gives

$$P = gm. \quad (178)$$

Combining (176) and (178) recovers

$$P = \frac{g\tau}{\kappa}, \quad (179)$$

the pressure–optical-depth relation used in the numerical photospheric example.

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