



STELLAR INTERIORS

Hydrostatic Equilibrium, the Virial Theorem and Polytropic Stellar Models ($n = 1.5$ and $n = 3$)

Academic Research Paper

Theoretical Astrophysics — Step-by-Step Analytic Derivations, Vector Diagrams and Numerical Tables

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Stellar Interiors

September 2026

Abstract. Hydrostatic stellar structure is the simplest nontrivial boundary-value problem in astrophysics: gravity attempts to collapse a self-gravitating gas, whereas pressure gradients support it. This paper develops the Newtonian theory from local momentum conservation to the hydrostatic and mass-continuity equations, derives central-pressure estimates, and proves the scalar virial theorem in the form appropriate to a gaseous star. The virial theorem is then used to estimate the Kelvin–Helmholtz time and the characteristic solar temperature. We next introduce polytopic equations of state, $P = K\rho^{1+1/n}$, and derive the Lane–Emden equation by a complete nondimensionalization. Its analytic solutions for $n = 0, 1, 5$, central series solution, global integral identities, and numerical constants for the astrophysically important cases $n = 1.5$ and $n = 3$ are presented. The mass–radius relation is derived, emphasizing the exceptional $n = 3$ case for which the mass is independent of radius at fixed K . Applications include fully convective stars, non-relativistic and relativistic white dwarfs, and the Eddington standard model. The treatment stresses the physical content of each mathematical result and the assumptions under which the classical theory succeeds or fails.

Keywords: stellar interiors; hydrostatic equilibrium; virial theorem; Lane–Emden equation; polytopes; white dwarfs

1. Introduction

The theory of stellar interiors begins with a deceptively simple observation: a normal star is long-lived because, on average, every mass element is nearly at rest. Gravity is enormous on terrestrial scales, yet a solar-type star neither collapses on its dynamical time nor expands freely into space. Its internal pressure adjusts until the net force on each spherical shell is almost zero. This mechanical balance is called hydrostatic equilibrium. Together with mass conservation, an equation of state, and equations for energy transport and generation, it forms the backbone of stellar-structure theory [3, 4, 5, 11, 12].

Historically, hydrostatic equilibrium and the virial theorem were already central to nineteenth-century attempts to understand the Sun. Kelvin and Helmholtz proposed that solar luminosity might be powered by gravitational contraction. Although nuclear fusion ultimately solved the energy problem, the Kelvin–Helmholtz argument remains fundamental: it reveals the natural gravitational energy scale of a star and explains why pre-main-sequence stars shine before hydrogen ignition. The virial theorem, in turn, gives the remarkable result that a self-gravitating ideal gas has negative heat capacity: losing energy makes it hotter. This counterintuitive fact underlies star formation, stellar contraction, and core collapse in stellar systems [8].

A particularly transparent class of hydrostatic models is supplied by polytopes,

$$P = K\rho^{1+1/n}, \quad (1)$$

where K and n are constants. Polytopic models are not merely mathematical toys. They arise when the gas is convective and nearly adiabatic, when degeneracy pressure dominates in white dwarfs, and when radiation

pressure and homologous structure lead to the Eddington standard model. The associated Lane–Emden equation was introduced in the classical works of Lane and Emden [1, 2] and remains one of the canonical nonlinear boundary-value problems in astrophysics.

The aim of this paper is to present a complete derivation of these ideas at a level suitable for advanced undergraduates, graduate students, and researchers who want a compact but rigorous reference. We begin from continuum momentum conservation, derive hydrostatic equilibrium and mass continuity, estimate central pressures, and then prove the virial theorem directly from the hydrostatic equation. We then develop the Lane–Emden formalism, including analytic solutions, central expansions, numerical constants, and mass–radius scalings. Special attention is given to the cases $n = 1.5$ and $n = 3$, because they correspond respectively to fully convective stars and non-relativistic degenerate objects, and to radiation-dominated or relativistically degenerate configurations.

2. Physical and Mathematical Framework

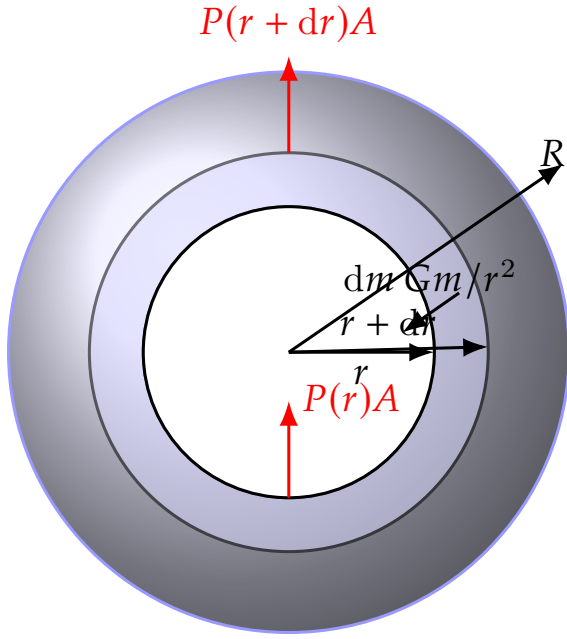
We consider an isolated, non-rotating, non-magnetic, Newtonian star in spherical symmetry. The independent radius is r , the density is $\rho(r)$, the pressure is $P(r)$, and the mass interior to r is $m(r)$. The gravitational potential Φ obeys Poisson’s equation,

$$\nabla^2\Phi = 4\pi G\rho. \quad (2)$$

In spherical symmetry the gravitational acceleration is

$$g(r) = -\frac{d\Phi}{dr} = -\frac{Gm(r)}{r^2}. \quad (3)$$

The sign convention is that r increases outward, while gravity points inward.



thin shell of area A and thickness dr

Figure 1. Free-body diagram for a thin spherical mass shell. The outward pressure force on the inner face exceeds the inward pressure force on the outer face by $-dP A$, and this difference balances the shell's weight in hydrostatic equilibrium.

Definition 1 (Hydrostatic stellar model). A hydrostatic stellar model is a time-independent spherical solution for which the fluid velocity vanishes, $\mathbf{v} = 0$, and pressure gradients exactly balance self-gravity.

The surface radius R is defined by

$$P(R) = 0, \quad m(R) = M, \quad (4)$$

where M is the total mass. The mean density is

$$\bar{\rho} = \frac{3M}{4\pi R^3}. \quad (5)$$

For a regular center,

$$m(0) = 0, \quad \frac{dP}{dr}(0) = 0, \quad \rho(0) = \rho_c < \infty. \quad (6)$$

3. Hydrostatic Equilibrium from Momentum Conservation

The Euler equation for an inviscid self-gravitating fluid is the local momentum balance

$$\rho \frac{d\mathbf{v}}{dt} = -\nabla P - \rho \nabla \Phi, \quad (7)$$

where the material derivative is

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla. \quad (8)$$

For a static star, $\mathbf{v} = 0$ and $\partial/\partial t = 0$, so (7) becomes

$$\nabla P = -\rho \nabla \Phi. \quad (9)$$

Taking the radial component and using (3) gives

$$\frac{dP}{dr} = -\rho \frac{d\Phi}{dr} = -\frac{Gm(r)\rho(r)}{r^2}. \quad (10)$$

This is the equation of hydrostatic equilibrium.

The same result follows from the shell force balance in Fig. 1. A shell of thickness dr and area A has mass

$$dm = \rho A dr. \quad (11)$$

The net outward pressure force is

$$F_P = P(r)A - P(r + dr)A. \quad (12)$$

Expanding to first order,

$$P(r + dr) = P(r) + \frac{dP}{dr} dr, \quad (13)$$

so

$$F_P = -A \frac{dP}{dr} dr. \quad (14)$$

The inward gravitational force is

$$F_g = -\frac{Gm(r)\rho A dr}{r^2}. \quad (15)$$

Static balance $F_P + F_g = 0$ yields

$$-A \frac{dP}{dr} dr - \frac{Gm\rho A dr}{r^2} = 0, \quad (16)$$

and therefore again (10).

The mass continuity equation follows from the definition of $m(r)$. The mass in a thin shell is

$$dm = 4\pi r^2 \rho(r) dr, \quad (17)$$

hence

$$\frac{dm}{dr} = 4\pi r^2 \rho. \quad (18)$$

Equations (10) and (18) form a closed mechanical system once an equation of state $P(\rho, \dots)$ is specified.

Remark 1. Equation (10) states that pressure must decrease outward. Since $m > 0$ and $\rho > 0$, $dP/dr < 0$ for $0 < r < R$. Thus the maximum pressure occurs at the center.

Combining (10) and (18) gives a second-order equation for P or ρ . From (10),

$$\frac{r^2}{\rho} \frac{dP}{dr} = -Gm. \quad (19)$$

Differentiating and substituting (18),

$$\frac{d}{dr} \left(\frac{r^2}{\rho} \frac{dP}{dr} \right) = -4\pi G r^2 \rho. \quad (20)$$

This is the spherical form of $\nabla \cdot (\rho^{-1} \nabla P) = -4\pi G \rho$.

4. Central Pressure Estimates and Bounds

A useful scale for the central pressure follows directly from dimensional analysis. Hydrostatic equilibrium gives

$$\frac{P_c}{R} \sim \frac{GM\rho}{R^2}. \quad (21)$$

With $\rho \sim M/R^3$,

$$P_c \sim \frac{GM^2}{R^4}. \quad (22)$$

For the Sun this scale is

$$\begin{aligned} \frac{GM_\odot^2}{R_\odot^4} &= \frac{(6.6743 \times 10^{-11})(1.9885 \times 10^{30})^2}{(6.957 \times 10^8)^4} \\ &= 1.13 \times 10^{15} \text{ Pa}. \end{aligned} \quad (23)$$

The actual solar central pressure is about 2.5×10^{16} Pa, larger by a factor of order 20 because the Sun is centrally condensed.

For an exactly uniform-density sphere,

$$\rho = \rho_0 = \frac{3M}{4\pi R^3}, \quad (24)$$

and

$$m(r) = \frac{4\pi}{3} \rho_0 r^3. \quad (25)$$

Hydrostatic equilibrium becomes

$$\frac{dP}{dr} = -\frac{4\pi}{3} G \rho_0^2 r. \quad (26)$$

Integrating from r to the surface, where $P(R) = 0$,

$$\begin{aligned} P(r) &= \int_r^R \frac{4\pi}{3} G \rho_0^2 r' dr' \\ &= \frac{2\pi}{3} G \rho_0^2 (R^2 - r^2). \end{aligned} \quad (27)$$

Thus the central pressure is

$$P_c = \frac{2\pi}{3} G \rho_0^2 R^2. \quad (28)$$

Substitution of (24) gives

$$P_c = \frac{3}{8\pi} \frac{GM^2}{R^4}. \quad (29)$$

Proposition 1 (Central-pressure lower estimate). *For a self-gravitating star of mass M and radius R , the minimum central pressure scale is of order the uniform-density value,*

$$P_c \gtrsim \frac{3}{8\pi} \frac{GM^2}{R^4}. \quad (30)$$

The physical reason is simple: if mass is more centrally concentrated than a uniform sphere, the gravitational acceleration in the inner star is larger where the overlying pressure must be supported. Central condensation therefore raises P_c above the uniform model. Equation (30) should be understood as a robust astrophysical lower estimate rather than an exact theorem for arbitrary pathological density distributions.

For the Sun, (29) gives

$$P_{c,\odot}^{\min} = \frac{3}{8\pi} \frac{GM_\odot^2}{R_\odot^4} = 1.35 \times 10^{14} \text{ Pa}. \quad (31)$$

This already exceeds atmospheric pressure by nine orders of magnitude, showing that stellar interiors are intrinsically high-pressure environments.

5. The Virial Theorem

The virial theorem relates internal thermal energy to gravitational binding energy. Begin with hydrostatic equilibrium,

$$\frac{dP}{dr} = -\frac{Gm\rho}{r^2}. \quad (32)$$

Multiply by $4\pi r^3 dr$:

$$4\pi r^3 \frac{dP}{dr} dr = -\frac{Gm}{r} 4\pi r^2 \rho dr. \quad (33)$$

Using $dm = 4\pi r^2 \rho dr$,

$$4\pi r^3 \frac{dP}{dr} dr = -\frac{Gm}{r} dm. \quad (34)$$

Integrating from center to surface,

$$\int_0^R 4\pi r^3 \frac{dP}{dr} dr = -\int_0^M \frac{Gm}{r} dm. \quad (35)$$

The right side is the gravitational potential energy,

$$\Omega = -\int_0^M \frac{Gm}{r} dm. \quad (36)$$

Integrate the left side by parts:

$$\int_0^R 4\pi r^3 \frac{dP}{dr} dr = [4\pi r^3 P]_0^R - \int_0^R 12\pi r^2 P dr. \quad (37)$$

Since $r^3 P \rightarrow 0$ at $r = 0$ and $P(R) = 0$,

$$[4\pi r^3 P]_0^R = 0. \quad (38)$$

Therefore

$$-3 \int_0^R 4\pi r^2 P dr = \Omega. \quad (39)$$

Equivalently,

$$3 \int P dV + \Omega = 0. \quad (40)$$

For a monatomic ideal gas,

$$P = \frac{\rho kT}{\mu m_H}, \quad (41)$$

and the thermal kinetic energy density is

$$u = \frac{3}{2} P. \quad (42)$$

The total thermal energy is

$$K = \int u \, dV = \frac{3}{2} \int P \, dV. \quad (43)$$

Substitution into (40) gives

$$2K + \Omega = 0. \quad (44)$$

Thus

$$K = -\frac{\Omega}{2}, \quad E = K + \Omega = \frac{\Omega}{2} = -K. \quad (45)$$

Remark 2. The total energy E of a bound ideal-gas star is negative. If the star radiates energy, E becomes more negative, so $K = -E$ increases. A self-gravitating ideal gas heats up as it loses energy.

The gravitational energy can be written as

$$\Omega = -\alpha \frac{GM^2}{R}, \quad (46)$$

where α depends on the density profile. For a uniform sphere,

$$\alpha = \frac{3}{5}. \quad (47)$$

For many stellar models α is of order unity.

6. Kelvin–Helmholtz Time and Mean Solar Temperature

If a star of luminosity L radiates by slowly losing gravitational energy, the characteristic Kelvin–Helmholtz time is

$$t_{\text{KH}} \sim \frac{|E|}{L} \sim \frac{GM^2}{RL}. \quad (48)$$

More precisely, using (45),

$$|E| = \frac{|\Omega|}{2} = \frac{\alpha}{2} \frac{GM^2}{R}. \quad (49)$$

Thus

$$t_{\text{KH}} = \frac{\alpha GM^2}{2RL}. \quad (50)$$

For the Sun, taking the conventional order-unity estimate without the factor $\alpha/2$,

$$\begin{aligned} t_{\text{KH},\odot} &\sim \frac{GM_{\odot}^2}{R_{\odot}L_{\odot}} \\ &= \frac{(6.6743 \times 10^{-11})(1.9885 \times 10^{30})^2}{(6.957 \times 10^8)(3.828 \times 10^{26})} \\ &= 9.91 \times 10^{14} \text{ s} = 3.14 \times 10^7 \text{ yr}. \end{aligned} \quad (51)$$

This is far shorter than the geological and nuclear age of the Sun, proving that gravitational contraction cannot power the present main-sequence Sun for 4.6×10^9 yr.

The virial theorem also estimates the mean solar temperature. Let the star contain

$$N = \frac{M}{\mu m_{\text{H}}} \quad (52)$$

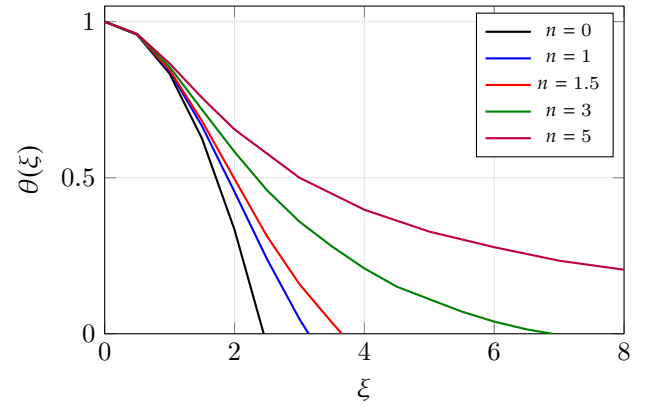


Figure 2. Lane–Emden functions for $n = 0, 1, 1.5, 3, 5$. The cases $n = 0, 1, 5$ are analytic; $n = 1.5$ and $n = 3$ are numerical. The $n = 5$ model approaches zero only as $\xi \rightarrow \infty$ and has infinite radius but finite mass.

particles, where μ is the mean molecular weight. Define a mass-weighted mean temperature $\bar{T}$ by

$$K = \frac{3}{2} N k \bar{T}. \quad (53)$$

Using $2K = |\Omega|$,

$$3Nk\bar{T} = \alpha \frac{GM^2}{R}. \quad (54)$$

Therefore

$$\bar{T} = \frac{\alpha}{3} \frac{GM\mu m_{\text{H}}}{kR}. \quad (55)$$

For the Sun, with $\mu \simeq 0.61$,

$$\begin{aligned} \frac{GM_{\odot}\mu m_{\text{H}}}{kR_{\odot}} &= \frac{(6.6743 \times 10^{-11})(1.9885 \times 10^{30})(0.61)(1.6735 \times 10^{-27})}{(1.3807 \times 10^{-23})(6.957 \times 10^8)} \\ &= 1.41 \times 10^7 \text{ K}. \end{aligned} \quad (56)$$

Thus

$$\bar{T}_{\odot} \simeq 4.7 \times 10^6 \alpha \text{ K}. \quad (57)$$

For $\alpha \simeq 1$, this is a few million kelvin, while the actual solar central temperature is $T_c \simeq 1.57 \times 10^7$ K. The difference reflects central concentration and the distinction between mean and central temperatures.

7. Polytropic Equations of State

Definition 2 (Polytropic equation of state). A polytropic gas satisfies

$$P = K\rho^{\gamma}, \quad \gamma = 1 + \frac{1}{n}, \quad (58)$$

where K is the polytropic constant and n is the polytropic index.

The index n measures compressibility. Smaller n gives a stiffer relation between pressure and density. Important cases include:

$$n = 0, \quad \rho = \text{constant}, \quad (59)$$

$$n = 1.5, \quad \gamma = 5/3, \quad (60)$$

$$n = 3, \quad \gamma = 4/3. \quad (61)$$

The value $n = 1.5$ describes an ideal monatomic gas in adiabatic convection and also a non-relativistic completely degenerate electron gas. The value $n = 3$ describes a radiation-pressure dominated Eddington model and an ultra-relativistic degenerate electron gas.

For a polytrope, hydrostatic equilibrium and mass continuity become

$$\frac{d}{dr} \left(\frac{r^2}{\rho} \frac{d}{dr} K \rho^{1+1/n} \right) = -4\pi G r^2 \rho. \quad (62)$$

The purpose of the Lane–Emden transformation is to remove dimensions and reduce this to a universal equation depending only on n .

8. Derivation of the Lane–Emden Equation

Introduce the dimensionless variables

$$\rho(r) = \rho_c \theta^n(\xi), \quad (63)$$

and

$$r = a\xi. \quad (64)$$

Then the pressure is

$$P = K \rho_c^{1+1/n} \theta^{n+1}. \quad (65)$$

Differentiate:

$$\begin{aligned} \frac{dP}{dr} &= K \rho_c^{1+1/n} (n+1) \theta^n \frac{d\theta}{dr} \\ &= (n+1) K \rho_c^{1/n} \rho_c \theta^n \frac{d\theta}{dr}. \end{aligned} \quad (66)$$

Since $\rho = \rho_c \theta^n$,

$$\frac{1}{\rho} \frac{dP}{dr} = (n+1) K \rho_c^{1/n} \frac{d\theta}{dr}. \quad (67)$$

Substitute this into (20):

$$(n+1) K \rho_c^{1/n} \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\theta}{dr} \right) = -4\pi G \rho_c \theta^n. \quad (68)$$

Now set $r = a\xi$, so

$$\frac{d}{dr} = \frac{1}{a} \frac{d}{d\xi}. \quad (69)$$

Also,

$$r^2 \frac{d\theta}{dr} = a \xi^2 \frac{d\theta}{d\xi}. \quad (70)$$

Therefore

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\theta}{dr} \right) = \frac{1}{a^2} \frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi} \right). \quad (71)$$

Choose

$$a^2 = \frac{(n+1)K}{4\pi G} \rho_c^{1/n-1}. \quad (72)$$

Then (68) reduces to

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi} \right) = -\theta^n. \quad (73)$$

Equivalently,

$$\frac{d^2\theta}{d\xi^2} + \frac{2}{\xi} \frac{d\theta}{d\xi} + \theta^n = 0. \quad (74)$$

The regular central boundary conditions are

$$\theta(0) = 1, \quad \theta'(0) = 0. \quad (75)$$

The stellar surface is at the first zero,

$$\theta(\xi_1) = 0, \quad R = a\xi_1. \quad (76)$$

Remark 3. The Lane–Emden equation is universal: for a fixed n , all dimensional models differ only by the scale factors a , ρ_c , and K .

The mass follows from (18):

$$\begin{aligned} m(r) &= 4\pi \int_0^r r'^2 \rho(r') dr' \\ &= 4\pi a^3 \rho_c \int_0^\xi \xi'^2 \theta^n d\xi'. \end{aligned} \quad (77)$$

Using (73),

$$\frac{d}{d\xi} (\xi^2 \theta') = -\xi^2 \theta^n. \quad (78)$$

Integrating from 0 to ξ ,

$$\int_0^\xi \xi'^2 \theta^n d\xi' = -\xi^2 \theta'. \quad (79)$$

Hence

$$m(\xi) = 4\pi a^3 \rho_c [-\xi^2 \theta'(\xi)]. \quad (80)$$

The total mass is

$$M = 4\pi a^3 \rho_c \omega_n, \quad (81)$$

where

$$\omega_n = -\xi_1^2 \theta'(\xi_1). \quad (82)$$

The mean density is

$$\bar{\rho} = \frac{3M}{4\pi R^3} = 3\rho_c \frac{\omega_n}{\xi_1^3}. \quad (83)$$

Therefore

$$\frac{\rho_c}{\bar{\rho}} = \frac{\xi_1^3}{3\omega_n}. \quad (84)$$

9. Analytic Lane–Emden Solutions

For $n = 0$, equation (74) becomes

$$\frac{1}{\xi^2} \frac{d}{d\xi} (\xi^2 \theta') = -1. \quad (85)$$

Multiplying by ξ^2 and integrating,

$$\xi^2 \theta' = -\frac{\xi^3}{3} + C. \quad (86)$$

Regularity requires $C = 0$, so

$$\theta' = -\frac{\xi}{3}. \quad (87)$$

A second integration with $\theta(0) = 1$ gives

$$\theta = 1 - \frac{\xi^2}{6}. \quad (88)$$

The first zero is

$$\xi_1 = \sqrt{6}. \quad (89)$$

The mass constant is

$$\omega_0 = -\xi_1^2 \theta'(\xi_1) = 2\sqrt{6}. \quad (90)$$

For $n = 1$, the Lane–Emden equation is linear:

$$\theta'' + \frac{2}{\xi} \theta' + \theta = 0. \quad (91)$$

Let

$$u = \xi \theta. \quad (92)$$

Then

$$u'' + u = 0. \quad (93)$$

The regular solution is

$$u = \sin \xi, \quad (94)$$

so

$$\theta = \frac{\sin \xi}{\xi}. \quad (95)$$

At the origin this is understood as the limiting value $\theta(0) = 1$. The first zero is

$$\xi_1 = \pi, \quad (96)$$

and

$$\omega_1 = \pi. \quad (97)$$

For $n = 5$, one may verify by substitution that

$$\theta = \left(1 + \frac{\xi^2}{3}\right)^{-1/2} \quad (98)$$

solves the Lane–Emden equation. Differentiation gives

$$\theta' = -\frac{\xi}{3} \left(1 + \frac{\xi^2}{3}\right)^{-3/2}. \quad (99)$$

Although θ has no finite zero,

$$\lim_{\xi \rightarrow \infty} \theta = 0, \quad (100)$$

the mass is finite because

$$\lim_{\xi \rightarrow \infty} [-\xi^2 \theta'] = \sqrt{3}. \quad (101)$$

Thus the $n = 5$ polytrope has infinite radius but finite mass.

10. Series Expansion Near the Center

Numerical integration cannot start exactly at $\xi = 0$ because of the apparent singular term $2\theta'/\xi$. Regularity is imposed by a power series,

$$\theta(\xi) = 1 + a_2 \xi^2 + a_4 \xi^4 + a_6 \xi^6 + \dots \quad (102)$$

Then

$$\theta' = 2a_2 \xi + 4a_4 \xi^3 + 6a_6 \xi^5 + \dots, \quad (103)$$

and

$$\theta'' = 2a_2 + 12a_4 \xi^2 + 30a_6 \xi^4 + \dots \quad (104)$$

The derivative terms in (74) give

$$\theta'' + \frac{2}{\xi} \theta' = 6a_2 + 20a_4 \xi^2 + 42a_6 \xi^4 + \dots \quad (105)$$

Expand the nonlinear term:

$$\begin{aligned} \theta^n &= (1 + a_2 \xi^2 + a_4 \xi^4 + \dots)^n \\ &= 1 + na_2 \xi^2 + \left[na_4 + \frac{n(n-1)}{2} a_2^2 \right] \xi^4 + \dots \end{aligned} \quad (106)$$

Equating powers in (74) gives

$$6a_2 + 1 = 0, \quad (107)$$

so

$$a_2 = -\frac{1}{6}. \quad (108)$$

At order ξ^2 ,

$$20a_4 + na_2 = 0, \quad (109)$$

hence

$$a_4 = \frac{n}{120}. \quad (110)$$

At order ξ^4 ,

$$42a_6 + na_4 + \frac{n(n-1)}{2} a_2^2 = 0. \quad (111)$$

Substituting (108) and (110),

$$a_6 = -\frac{n(8n-5)}{15120}. \quad (112)$$

Thus the regular central expansion is

$$\theta = 1 - \frac{\xi^2}{6} + \frac{n\xi^4}{120} - \frac{n(8n-5)\xi^6}{15120} + \dots \quad (113)$$

11. Numerical Polytropic Constants

The two most important non-analytic cases here are $n = 1.5$ and $n = 3$. Direct numerical integration of (74), initialized with (113), gives

$$\xi_1(n = 1.5) = 3.65375, \quad (114)$$

$$\omega_{1.5} = 2.71406, \quad (115)$$

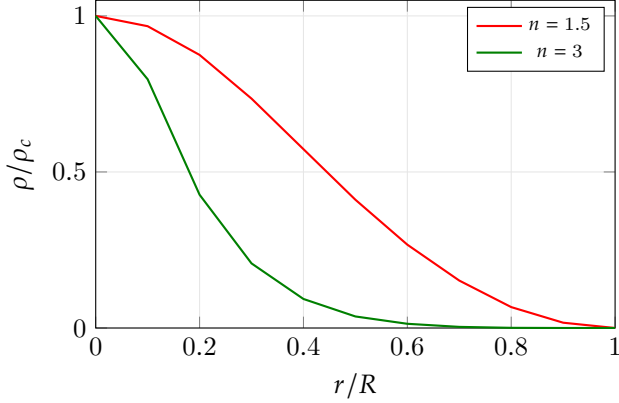
and

$$\xi_1(n = 3) = 6.89685, \quad (116)$$

$$\omega_3 = 2.01824. \quad (117)$$

Table 1. Selected Lane–Emden constants. Here ξ_1 is the first zero, $\omega_n = -\xi_1^2 \theta'(\xi_1)$, and $\rho_c/\bar{\rho} = \xi_1^3/(3\omega_n)$.

n	ξ_1	ω_n	$\rho_c/\bar{\rho}$
0	2.44949	4.89898	1.00
1	3.14159	3.14159	3.29
1.5	3.65375	2.71406	5.99
3	6.89685	2.01824	54.18
5	∞	1.73205	0

**Figure 3.** Normalized density profiles for $n = 1.5$ and $n = 3$. The $n = 3$ model places much more mass near the center, explaining its large $\rho_c/\bar{\rho}$.

Using (84),

$$\frac{\rho_c}{\bar{\rho}}(n = 1.5) = \frac{(3.65375)^3}{3(2.71406)} = 5.99, \quad (118)$$

and

$$\frac{\rho_c}{\bar{\rho}}(n = 3) = \frac{(6.89685)^3}{3(2.01824)} = 54.18. \quad (119)$$

The $n = 3$ model is therefore far more centrally condensed than the $n = 1.5$ model.

12. Mass–Radius Relation

The dimensional radius is

$$R = a\xi_1, \quad (120)$$

with

$$a = \left[\frac{(n+1)K}{4\pi G} \right]^{1/2} \rho_c^{(1-n)/(2n)}. \quad (121)$$

Therefore

$$R = \xi_1 \left[\frac{(n+1)K}{4\pi G} \right]^{1/2} \rho_c^{(1-n)/(2n)}. \quad (122)$$

Similarly,

$$M = 4\pi\omega_n \left[\frac{(n+1)K}{4\pi G} \right]^{3/2} \rho_c^{(3-n)/(2n)}. \quad (123)$$

Eliminate ρ_c . From (122),

$$\rho_c^{(1-n)/(2n)} = \frac{R}{\xi_1} \left[\frac{4\pi G}{(n+1)K} \right]^{1/2}. \quad (124)$$

Raise to the power $(3-n)/(1-n)$:

$$\rho_c^{(3-n)/(2n)} = \left[\frac{R}{\xi_1} \left(\frac{4\pi G}{(n+1)K} \right)^{1/2} \right]^{(3-n)/(1-n)}. \quad (125)$$

Substitution in (123) yields

$$M = C_n K^{n/(n-1)} R^{(3-n)/(1-n)}, \quad (126)$$

where

$$C_n = 4\pi\omega_n \left[\frac{n+1}{4\pi G} \right]^{n/(n-1)} \xi_1^{(n-3)/(1-n)}. \quad (127)$$

Equivalently,

$$R \propto K^{n/(3-n)} M^{(1-n)/(3-n)} \quad (n \neq 3). \quad (128)$$

The special case $n = 3$ follows directly from (123). Since the exponent of ρ_c is zero,

$$M = 4\pi\omega_3 \left[\frac{4K}{4\pi G} \right]^{3/2}, \quad n = 3. \quad (129)$$

Thus the mass is independent of radius and central density at fixed K . This mathematical fact is the origin of the Chandrasekhar limiting mass for relativistically degenerate white dwarfs.

For $n = 1.5$, (128) becomes

$$R \propto K M^{-1/3}, \quad (130)$$

or equivalently

$$M \propto K^3 R^{-3}. \quad (131)$$

This inverse mass–radius relation is the hallmark of non-relativistic white dwarfs: adding mass makes the star smaller.

13. Worked Numerical Applications

13.1. Solar hydrostatic scales

Using (5), the solar mean density is

$$\begin{aligned} \bar{\rho}_\odot &= \frac{3M_\odot}{4\pi R_\odot^3} \\ &= \frac{3(1.9885 \times 10^{30})}{4\pi(6.957 \times 10^8)^3} \\ &= 1.41 \times 10^3 \text{ kg m}^{-3}. \end{aligned} \quad (132)$$

The free-fall or dynamical time is

$$t_{\text{dyn}} \sim (G\bar{\rho})^{-1/2}. \quad (133)$$

For the Sun,

$$t_{\text{dyn},\odot} = [(6.6743 \times 10^{-11})(1.41 \times 10^3)]^{-1/2}$$

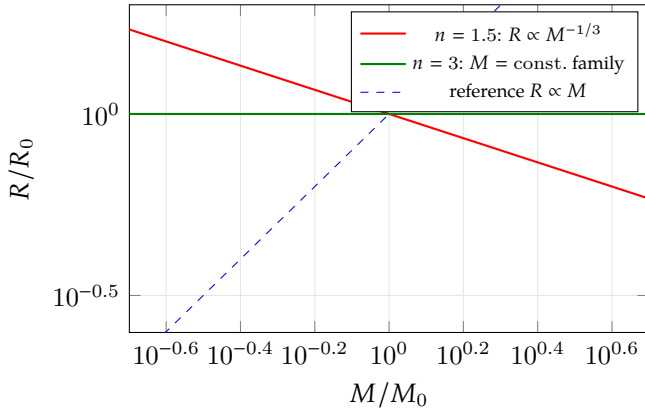


Figure 4. Mass–radius scalings at fixed polytropic constant. The $n = 1.5$ curve is appropriate to non-relativistic degeneracy. The $n = 3$ case is singular: at fixed K the mass is fixed while the radius changes with central density.

$$= 3.26 \times 10^3 \text{ s}. \quad (134)$$

This is about one hour, vastly shorter than the solar age. Hydrostatic equilibrium is therefore an excellent approximation for main-sequence evolution.

If the Sun were approximated by an $n = 3$ polytrope, its central density would be

$$\rho_c = 54.18 \bar{\rho}_\odot = 7.64 \times 10^4 \text{ kg m}^{-3}. \quad (135)$$

This is close in order of magnitude to detailed solar-model values.

13.2. Fully convective stars

In a chemically homogeneous convective region, the stratification is nearly adiabatic:

$$P \rho^{-\gamma_{\text{ad}}} = \text{constant}. \quad (136)$$

For a monatomic ideal gas,

$$\gamma_{\text{ad}} = \frac{5}{3}. \quad (137)$$

Thus

$$1 + \frac{1}{n} = \frac{5}{3}, \quad (138)$$

so

$$n = \frac{3}{2}. \quad (139)$$

Low-mass main-sequence stars and pre-main-sequence Hayashi-track stars are therefore often well represented by $n = 1.5$ polytropes. Their relatively small central concentration,

$$\frac{\rho_c}{\bar{\rho}} = 5.99, \quad (140)$$

reflects efficient convective entropy mixing.

13.3. Non-relativistic white dwarfs

A completely degenerate, non-relativistic electron gas has pressure

$$P = K_{\text{NR}} \rho^{5/3}, \quad (141)$$

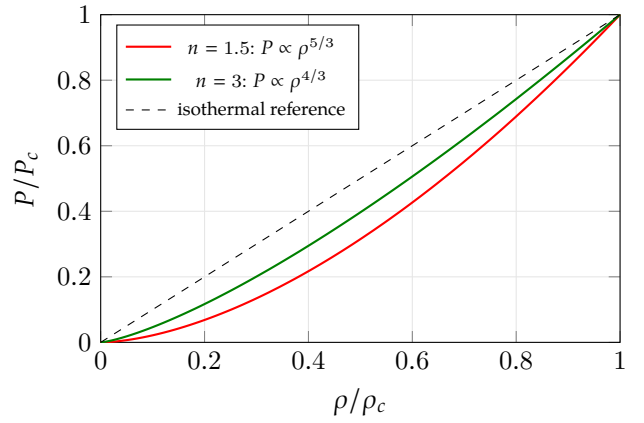


Figure 5. Normalized polytropic equations of state. The $n = 1.5$ relation is stiffer than the $n = 3$ relation, so $n = 1.5$ configurations are less centrally condensed.

where

$$K_{\text{NR}} = \frac{\hbar^2}{5m_e} (3\pi^2)^{2/3} \frac{1}{(\mu_e m_{\text{H}})^{5/3}}. \quad (142)$$

This is an $n = 1.5$ polytrope. Hence

$$R \propto M^{-1/3}. \quad (143)$$

The physics is transparent: increasing M increases gravity; the star responds by compressing the electron gas, raising the Fermi momentum and degeneracy pressure. Compression rather than heating supplies the support, so a more massive non-relativistic white dwarf is smaller.

13.4. Relativistic degeneracy and the Chandrasekhar scale

For ultra-relativistic degenerate electrons,

$$P = K_{\text{R}} \rho^{4/3}, \quad (144)$$

with

$$K_{\text{R}} = \frac{\hbar c}{4} (3\pi^2)^{1/3} \frac{1}{(\mu_e m_{\text{H}})^{4/3}}. \quad (145)$$

This is an $n = 3$ polytrope. Equation (129) implies a limiting mass,

$$M_{\text{Ch}} \propto K_{\text{R}}^{3/2} G^{-3/2}. \quad (146)$$

A full treatment gives

$$M_{\text{Ch}} \simeq \frac{5.83}{\mu_e^2} M_\odot. \quad (147)$$

For carbon–oxygen matter, $\mu_e \simeq 2$, so

$$M_{\text{Ch}} \simeq 1.46 M_\odot. \quad (148)$$

The polytropic singularity at $n = 3$ is therefore not a mathematical curiosity; it marks the boundary beyond which cold electron degeneracy cannot stabilize a Newtonian star.

14. Discussion

The theory developed above is powerful because it isolates the mechanical consequences of self-gravity. Hydrostatic equilibrium, (10), contains no reference to the microscopic origin of pressure. Gas pressure, radiation pressure, degeneracy pressure, turbulent pressure, and magnetic stresses can all be incorporated by changing the stress law. The polytropic approximation is successful whenever the pressure can be expressed as a simple power of density with nearly constant entropy or fixed composition.

Its limitations are equally important. Real stars are not exact polytropes. They have composition gradients, ionization zones, nuclear energy sources, opacity-dependent radiative transport, convection, rotation, magnetic fields, and mass loss. The Sun, for example, is radiative through much of its interior and convective only in its outer envelope. A single $n = 3$ polytrope captures its central concentration better than a uniform model, but detailed helioseismic solar models require the full stellar-structure equations with realistic microphysics [10, 11].

The virial theorem is more general than hydrostatic equilibrium. In tensor form it applies to collisionless stellar systems, molecular clouds, and dark-matter halos [8]. Surface pressure, rotation, magnetic fields, and time-dependent inertia add terms to (40). For example, a pressure-confined molecular cloud satisfies

$$3 \int P dV + \Omega = 3P_s V, \quad (149)$$

where P_s is the external surface pressure. A rotating star has an additional positive rotational kinetic energy term, reducing the pressure support required at fixed mass and radius.

The $n = 3$ case deserves special emphasis. Its mass independence at fixed K follows from scale invariance: for $\gamma = 4/3$, pressure and gravity scale identically under homologous compression. This is why radiation-dominated massive stars are close to marginal dynamical stability and why relativistic degeneracy leads to the Chandrasekhar mass. General relativity, inverse beta decay, Coulomb corrections, and finite-temperature effects modify the precise limiting mass, but the Newtonian $n = 3$ polytrope identifies the controlling physics [4, 7].

For $n = 1.5$, the situation is more stable. The equation of state is stiffer, $\gamma = 5/3$, so compression increases pressure faster than required by homologous gravity. Fully convective stars and brown dwarfs therefore possess relatively broad density profiles, while non-relativistic white dwarfs follow the robust inverse mass-radius law. Observed white-dwarf radii broadly confirm this trend, with deviations caused by finite temperature, envelope composition, crystallization, and relativistic corrections.

Modern stellar evolution codes solve the same mechanical equations derived here, but coupled to nuclear

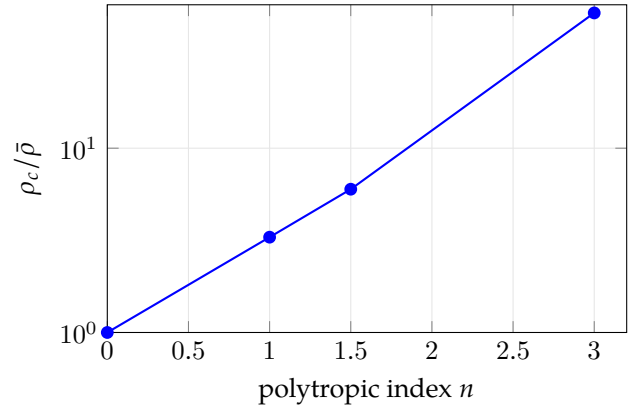


Figure 6. Growth of central concentration with polytropic index. Larger n corresponds to a softer equation of state and hence stronger central condensation.

reaction networks, opacity tables, equations of state, diffusion, convection prescriptions, and atmosphere boundary conditions. Nevertheless, the polytropic solutions remain indispensable. They provide analytic checks, scaling relations, initial models, and physical intuition. Their constants, such as ξ_1 , ω_n , and $\rho_c / \bar{\rho}$, are still encountered throughout compact-object theory, stellar pulsation theory, and star-formation studies.

15. Conclusion

Starting from Newtonian momentum conservation, we derived the hydrostatic equation

$$\frac{dP}{dr} = -\frac{Gm\rho}{r^2} \quad (150)$$

and the mass-continuity equation

$$\frac{dm}{dr} = 4\pi r^2 \rho. \quad (151)$$

These equations imply central pressures of order GM^2/R^4 and show why stellar interiors are extreme high-pressure systems.

By multiplying hydrostatic equilibrium by $4\pi r^3$ and integrating, we obtained the scalar virial theorem,

$$3 \int P dV + \Omega = 0, \quad (152)$$

and for a monatomic ideal gas,

$$2K + \Omega = 0. \quad (153)$$

This explains the Kelvin–Helmholtz time, the gravitational temperature scale, and the negative heat capacity of self-gravitating gas.

For polytropes, $P = K\rho^{1+1/n}$, nondimensionalization led to the Lane–Emden equation,

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi} \right) = -\theta^n, \quad (154)$$

with $\theta(0) = 1$ and $\theta'(0) = 0$. The analytic cases $n = 0, 1, 5$ and the numerical cases $n = 1.5, 3$ reveal how compressibility controls central condensation. The constants

$$\xi_1 = 3.65375, \quad \omega_{1.5} = 2.71406 \quad (155)$$

and

$$\xi_1 = 6.89685, \quad \omega_3 = 2.01824 \quad (156)$$

give $\rho_c/\bar{\rho} = 5.99$ and 54.18 , respectively.

Finally, the mass–radius relation

$$M \propto K^{n/(n-1)} R^{(3-n)/(1-n)} \quad (157)$$

shows that $n = 1.5$ stars obey $R \propto M^{-1/3}$ at fixed K , while $n = 3$ stars have mass independent of radius. These two cases underlie the physics of convective stars, white dwarfs, radiation-dominated stars, and the Chandrasekhar mass. The classical theory is therefore not merely pedagogical; it is a compact expression of the dominant balances that organize much of stellar astrophysics.

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