



DEGENERATE MATTER

Relativistic Electron Degeneracy Pressure and the Analytic Derivation of the Chandrasekhar Mass Limit

Academic Research Paper

Theoretical Astrophysics — Step-by-Step Analytic Derivations, Vector Diagrams and Numerical Tables

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Abstract. White dwarfs are supported not by thermal pressure but by the quantum pressure of a cold, degenerate electron gas. Their maximum mass is therefore one of the cleanest macroscopic consequences of the Pauli exclusion principle, special relativity, and Newtonian self-gravity. This paper gives a self-contained derivation of the electron degeneracy equation of state and the Chandrasekhar mass limit. Beginning with phase-space counting in a Fermi sphere, we derive the Fermi momentum, the exact zero-temperature pressure integral, its non-relativistic and ultra-relativistic limits, and Chandrasekhar's closed equation of state in terms of a relativity parameter $x = p_F/(m_e c)$. We then combine these results with hydrostatic equilibrium and the Lane–Emden equation. The non-relativistic $n = 3/2$ polytrope yields the white-dwarf mass–radius law $R \propto M^{-1/3}$, while the ultra-relativistic $n = 3$ polytrope yields a radius-independent limiting mass. Using the Lane–Emden constant $\xi_1^2 |\theta'(\xi_1)| = 2.01824$, we obtain $M_{\text{Ch}} = 5.83 \mu_e^{-2} M_\odot$, or $1.457 M_\odot$ for carbon–oxygen matter with $\mu_e = 2$. We also discuss numerical examples, observational tests, finite-temperature scales, stability criteria, and physical corrections.

Keywords: white dwarfs; degenerate matter; Chandrasekhar limit; Fermi gas; polytropes; Type Ia supernovae

1. Introduction

The existence of white dwarfs shows that stars may reach an equilibrium in which temperature is dynamically secondary. Sirius B, discovered as a faint companion to Sirius and later recognized as compact and hot, presented an early paradox: its luminosity and spectrum implied a stellar temperature, but its gravitational mass and small radius implied mean densities many orders of magnitude above terrestrial matter. The resolution was quantum mechanical. Fowler [2] applied Fermi–Dirac statistics to dense stellar matter, showing that an electron gas at sufficiently high density becomes degenerate and exerts pressure even at zero temperature. Stoner [3] and Chandrasekhar [4, 5] then incorporated special relativity and found that electron degeneracy pressure has a limiting ability to support matter against gravity. The result is the Chandrasekhar mass, a maximum mass of order $1.4 M_\odot$ for cold carbon–oxygen white dwarfs. Standard treatments are given in [1, 6, 10, 11].

The physics is unusually transparent. Pauli exclusion permits at most two electrons, one for each spin state, in a phase-space cell of volume h^3 . Compression therefore forces electrons into states of larger momentum. For non-relativistic electrons, the kinetic energy scales as $p^2/(2m_e)$, and the pressure scales as $n_e^{5/3}$. In that regime the star is described by a polytrope of index $n = 3/2$ and the radius decreases as $M^{-1/3}$. At still higher density, electrons become relativistic, $E \simeq pc$, and the pressure scales only as $n_e^{4/3}$. This softer equation of state corresponds to a polytrope of index $n = 3$, for which the mass is independent of central density and radius. Increasing central density then no longer increases the supported mass; it merely reduces the radius. The limiting mass is fixed by $\hbar$, c , G , and the

electron-per-baryon ratio.

The same limit has broad astrophysical significance. It marks the upper mass of stable cold white dwarfs, explains why observed white-dwarf masses cluster well below the maximum, and underlies the thermonuclear instability of Type Ia supernova progenitors [14]. It is not, however, an exact universal number. Inverse beta decay, Coulomb terms, finite-temperature corrections, general relativity, rotation, crystallization, and composition all shift the ideal Newtonian value. The clean analytic derivation remains fundamental because it identifies the governing dimensionless competition between relativistic Fermi pressure and self-gravity.

This paper is written as a rigorous derivation rather than a historical review. We first define the phase-space measure and derive the zero-temperature Fermi sphere. We then compute the pressure integral exactly and recover the limiting power laws. Finally, we solve the hydrostatic problem using polytropic theory and evaluate the numerical constants explicitly.

2. Physical and Mathematical Framework

2.1. Pauli exclusion and the cold Fermi sea

Definition 1 (Complete electron degeneracy). An electron gas is completely degenerate when the temperature satisfies

$$k_B T \ll E_F, \quad (1)$$

where E_F is the Fermi energy. At zero temperature all one-particle momentum states with $p < p_F$ are occupied and all states with $p > p_F$ are empty.

For an electron gas in a volume V , the number of

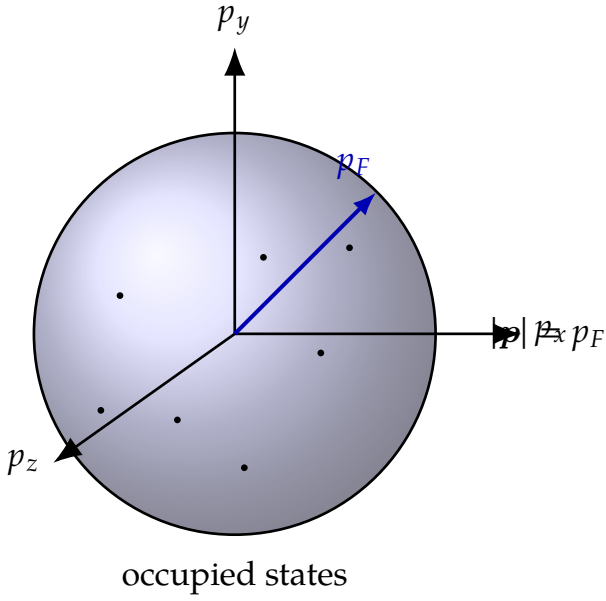


Figure 1. The zero-temperature Fermi sphere in momentum space. Pauli exclusion fills all electron states inside a sphere of radius p_F , with two spin states per cell h^3 .

quantum states in an element $d^3x d^3p$ is

$$dN_{\text{states}} = g_s \frac{d^3x d^3p}{h^3}, \quad (2)$$

where $g_s = 2$ counts spin degeneracy. At $T = 0$ the occupied region in momentum space is a sphere of radius p_F , shown in Fig. 1. Therefore

$$\begin{aligned} N_e &= 2 \frac{V}{h^3} \int_{p < p_F} d^3p \\ &= 2 \frac{V}{h^3} 4\pi \int_0^{p_F} p^2 dp \\ &= \frac{8\pi V}{3h^3} p_F^3. \end{aligned} \quad (3)$$

Dividing by V gives

$$n_e = \frac{8\pi}{3h^3} p_F^3 = \frac{p_F^3}{3\pi^2 \hbar^3}, \quad (4)$$

since $h = 2\pi\hbar$. Solving (4) yields the central kinematic scale:

$$p_F = \hbar(3\pi^2 n_e)^{1/3}. \quad (5)$$

Remark 1. Equation (5) is independent of the electron mass. It is purely a consequence of quantum state counting. The mass enters only when momenta are converted into velocities and energies.

2.2. Composition and the mean molecular weight per electron

The mass density is carried almost entirely by nuclei, while the pressure is dominated by electrons. If the material contains nuclear species i with charge Z_i , mass

number A_i , and mass fraction X_i , then the electron number density is

$$n_e = \rho \sum_i \frac{Z_i X_i}{A_i m_u}, \quad (6)$$

where m_u is the atomic mass unit. It is conventional to define the mean molecular weight per electron by

$$\frac{1}{\mu_e} = \sum_i \frac{Z_i X_i}{A_i}. \quad (7)$$

Thus

$$n_e = \frac{\rho}{\mu_e m_u}. \quad (8)$$

For helium, carbon, oxygen, neon, and magnesium, $A/Z \approx 2$, so $\mu_e \approx 2$. This simple fact explains why the ideal Chandrasekhar mass is nearly composition-independent for ordinary carbon–oxygen and oxygen–neon white dwarfs.

Combining (5) and (8), the Fermi momentum may be written as

$$p_F = \hbar \left(\frac{3\pi^2 \rho}{\mu_e m_u} \right)^{1/3}. \quad (9)$$

The dimensionless relativity parameter is

$$x \equiv \frac{p_F}{m_e c}. \quad (10)$$

Equations (9) and (10) imply

$$\rho = \rho_0 \mu_e x^3, \quad \rho_0 = \frac{m_u}{3\pi^2} \left(\frac{m_e c}{\hbar} \right)^3. \quad (11)$$

Numerically,

$$\rho_0 = 9.74 \times 10^5 \text{ g cm}^{-3}. \quad (12)$$

For $\mu_e = 2$, electrons become relativistic, $x \approx 1$, near $\rho \approx 1.95 \times 10^6 \text{ g cm}^{-3}$.

3. Degenerate Electron Pressure

3.1. Kinetic pressure integral

For an isotropic gas, pressure is the flux of momentum through a unit area. A particle with momentum p and speed $v(p)$ contributes a normal momentum flux proportional to $p v \cos^2 \alpha$, where α is the angle between the momentum and the surface normal. Angular averaging gives

$$\langle \cos^2 \alpha \rangle = \frac{1}{3}. \quad (13)$$

Hence the zero-temperature pressure is

$$P = \frac{1}{3} \int_0^{p_F} p v(p) n(p) dp, \quad (14)$$

where $n(p) dp$ is the number density of electrons with momenta between p and $p + dp$. From (2),

$$n(p) dp = 2 \frac{4\pi p^2 dp}{h^3} = \frac{8\pi p^2}{h^3} dp. \quad (15)$$

Table 1. Constants used in the numerical evaluations.

Quantity	Symbol	Value
Planck constant	h	$6.62607015 \times 10^{-27}$ erg s
Reduced Planck constant	$\hbar$	$1.054571817 \times 10^{-27}$ erg s
Speed of light	c	$2.99792458 \times 10^{10}$ cm s $^{-1}$
Electron mass	m_e	$9.1093837 \times 10^{-28}$ g
Atomic mass unit	m_u	$1.6605391 \times 10^{-24}$ g
Gravitational constant	G	6.67430×10^{-8} cgs
Solar mass	$M_\odot$	1.98847×10^{33} g
Solar radius	$R_\odot$	6.957×10^{10} cm

Thus

$$P = \frac{1}{3} \int_0^{p_F} p v(p) \frac{8\pi p^2}{h^3} dp$$

$$= \frac{8\pi}{3h^3} \int_0^{p_F} p^3 v(p) dp. \quad (16)$$

The relativistic energy is

$$E(p) = \sqrt{p^2 c^2 + m_e^2 c^4}, \quad (17)$$

and the particle speed is the group velocity

$$v(p) = \frac{dE}{dp} = \frac{pc^2}{\sqrt{p^2 c^2 + m_e^2 c^4}}. \quad (18)$$

Substitution into (16) gives the exact cold pressure integral

$$P = \frac{8\pi c^2}{3h^3} \int_0^{p_F} \frac{p^4 dp}{\sqrt{p^2 c^2 + m_e^2 c^4}}. \quad (19)$$

The same expression follows from thermodynamics. If u_e is the electron energy density including rest mass, then at zero temperature

$$P = n_e^2 \frac{d}{dn_e} \left(\frac{u_e}{n_e} \right). \quad (20)$$

This form is useful because it emphasizes that the pressure is not an independent assumption: it is the response of the filled Fermi sea to compression. The energy density is

$$u_e = \frac{8\pi}{h^3} \int_0^{p_F} p^2 E(p) dp. \quad (21)$$

Since $n_e \propto p_F^3$, a compression changes both the number of occupied states per unit volume and the upper limit of the integral. Differentiating (21) with respect to p_F and using (4) recovers (19). This consistency check is important: the degeneracy pressure is the mechanical expression of the same ground-state energy that enters the virial theorem.

3.2. Non-relativistic limit

When $p_F \ll m_e c$, the velocity is

$$v(p) \simeq \frac{p}{m_e}. \quad (22)$$

Equation (16) becomes

$$P_{NR} = \frac{8\pi}{3h^3 m_e} \int_0^{p_F} p^4 dp$$

$$= \frac{8\pi}{15h^3 m_e} p_F^5. \quad (23)$$

Using $h = 2\pi\hbar$,

$$P_{NR} = \frac{p_F^5}{15\pi^2 \hbar^3 m_e}. \quad (24)$$

Now insert (5):

$$P_{NR} = \frac{1}{15\pi^2 \hbar^3 m_e} \hbar^5 (3\pi^2 n_e)^{5/3}$$

$$= \frac{\hbar^2}{5m_e} (3\pi^2)^{2/3} n_e^{5/3}. \quad (25)$$

In terms of mass density,

$$P_{NR} = K_{NR} \rho^{5/3}, \quad (26)$$

where

$$K_{NR} = \frac{\hbar^2}{5m_e} (3\pi^2)^{2/3} (\mu_e m_u)^{-5/3}. \quad (27)$$

Remark 2. The exponent 5/3 has a simple origin. Compression raises the characteristic momentum as $p_F \propto n_e^{1/3}$; non-relativistic kinetic energy scales as p_F^2 ; pressure is energy density and therefore scales as $n_e p_F^2 \propto n_e^{5/3}$.

3.3. Ultra-relativistic limit

When $p_F \gg m_e c$, the electron speed tends to

$$v(p) \simeq c. \quad (28)$$

Equation (16) gives

$$P_{UR} = \frac{8\pi c}{3h^3} \int_0^{p_F} p^3 dp$$

$$= \frac{2\pi c}{3h^3} p_F^4. \quad (29)$$

Equivalently,

$$P_{UR} = \frac{c}{12\pi^2 \hbar^3} p_F^4. \quad (30)$$

Using (5),

$$\begin{aligned} P_{\text{UR}} &= \frac{c}{12\pi^2\hbar^3} \hbar^4 (3\pi^2 n_e)^{4/3} \\ &= \frac{\hbar c}{4} (3\pi^2)^{1/3} n_e^{4/3}. \end{aligned} \quad (31)$$

Thus

$$P_{\text{UR}} = K_{\text{UR}} \rho^{4/3}, \quad (32)$$

with

$$K_{\text{UR}} = \frac{\hbar c}{4} (3\pi^2)^{1/3} (\mu_e m_u)^{-4/3}. \quad (33)$$

Remark 3. Relativity softens the equation of state. Since $E \simeq pc$ rather than $p^2/(2m_e)$, pressure scales as $n_e p_F \propto n_e^{4/3}$. This change from 5/3 to 4/3 is the mathematical source of the limiting mass.

3.4. Chandrasekhar's exact zero-temperature equation of state

Set

$$p = m_e c y, \quad p_F = m_e c x. \quad (34)$$

Then (19) becomes

$$\begin{aligned} P &= \frac{8\pi m_e^4 c^5}{3\hbar^3} \int_0^x \frac{y^4 dy}{\sqrt{1+y^2}} \\ &= \frac{m_e^4 c^5}{3\pi^2 \hbar^3} \int_0^x \frac{y^4 dy}{\sqrt{1+y^2}}. \end{aligned} \quad (35)$$

The integral has the closed form

$$\int_0^x \frac{y^4 dy}{\sqrt{1+y^2}} = \frac{1}{8} \left[x(2x^2 - 3)\sqrt{1+x^2} + 3 \sinh^{-1} x \right]. \quad (36)$$

Therefore

$$P = A f(x), \quad (37)$$

where

$$A = \frac{m_e^4 c^5}{24\pi^2 \hbar^3} \quad (38)$$

and

$$f(x) = x(2x^2 - 3)(1+x^2)^{1/2} + 3 \sinh^{-1} x. \quad (39)$$

The associated density is

$$\rho = \rho_0 \mu_e x^3, \quad (40)$$

with ρ_0 defined in (11).

The logarithmic stiffness of the exact equation of state is measured by

$$\gamma(x) = \frac{d \ln P}{d \ln \rho}. \quad (41)$$

Since $\rho \propto x^3$, differentiation of (39) gives

$$\gamma(x) = \frac{x f'(x)}{3 f(x)} = \frac{8x^5}{3(1+x^2)^{1/2} f(x)}. \quad (42)$$

Thus

$$\lim_{x \rightarrow 0} \gamma(x) = \frac{5}{3}, \quad \lim_{x \rightarrow \infty} \gamma(x) = \frac{4}{3}. \quad (43)$$

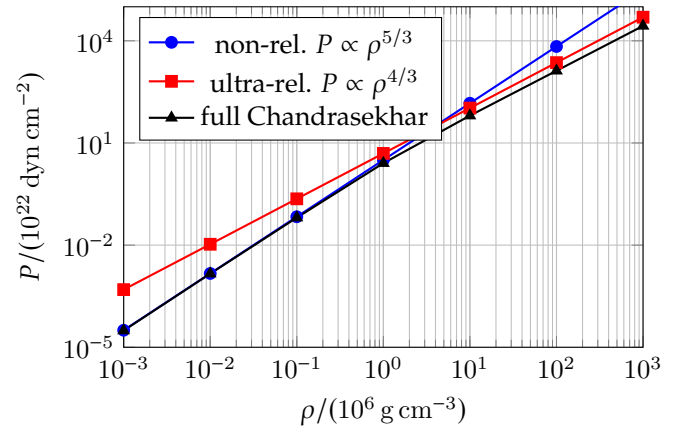


Figure 2. Degenerate-electron pressure for $\mu_e = 2$. The full Chandrasekhar equation of state approaches the non-relativistic curve at low density and the ultra-relativistic curve at high density.

The approach to 4/3 is gradual: massive white dwarfs are not exactly $n = 3$ polytropes throughout their interiors, but their central regions become increasingly close to this marginal stiffness.

The limiting forms follow by expansion. For $x \ll 1$,

$$f(x) = \frac{8}{5} x^5 - \frac{4}{7} x^7 + O(x^9), \quad (44)$$

so that (37) reduces to (25). For $x \gg 1$,

$$f(x) = 2x^4 - 2x^2 + 3 \ln(2x) + O(x^{-2}), \quad (45)$$

and the leading term gives (31). Figure 2 displays the transition.

4. Hydrostatic Equilibrium and Polytropes

4.1. Newtonian stellar structure

For a static, spherical star, mass conservation and hydrostatic balance are

$$\frac{dm}{dr} = 4\pi r^2 \rho, \quad (46)$$

and

$$\frac{dP}{dr} = -\frac{Gm(r)\rho}{r^2}. \quad (47)$$

A polytropic equation of state has the form

$$P = K \rho^{1+1/n}, \quad (48)$$

where n is the polytropic index. Introduce

$$\rho(r) = \rho_c \theta^n(\xi), \quad r = a \xi, \quad (49)$$

with

$$a^2 = \frac{(n+1)K}{4\pi G} \rho_c^{1/n-1}. \quad (50)$$

The stellar structure equations reduce to the Lane-Emden equation,

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi} \right) = -\theta^n, \quad (51)$$

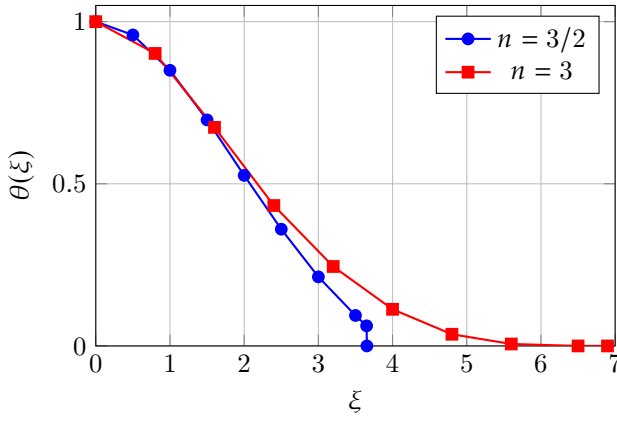


Figure 3. Representative Lane-Emden functions for non-relativistic and ultra-relativistic degenerate white dwarfs. The first zero fixes the surface.

with central boundary conditions

$$\theta(0) = 1, \quad \theta'(0) = 0. \quad (52)$$

The stellar surface is the first zero,

$$\theta(\xi_1) = 0. \quad (53)$$

The radius is

$$R = a\xi_1, \quad (54)$$

and the mass is

$$\begin{aligned} M &= 4\pi \int_0^R r^2 \rho \, dr \\ &= 4\pi a^3 \rho_c \int_0^{\xi_1} \xi^2 \theta^n \, d\xi. \end{aligned} \quad (55)$$

Using (51),

$$\int_0^{\xi_1} \xi^2 \theta^n \, d\xi = -\xi_1^2 \theta'(\xi_1), \quad (56)$$

so

$$M = 4\pi a^3 \rho_c \xi_1^2 |\theta'(\xi_1)|. \quad (57)$$

The central-to-mean density ratio of a polytrope is also fixed by the Lane-Emden solution. With

$$\bar{\rho} = \frac{3M}{4\pi R^3}, \quad (58)$$

equations (54) and (57) imply

$$\frac{\rho_c}{\bar{\rho}} = \frac{\xi_1}{3|\theta'(\xi_1)|}. \quad (59)$$

This ratio is 5.99 for $n = 3/2$ and 54.2 for $n = 3$. Relativistic white dwarfs are therefore much more centrally condensed than low-mass non-relativistic white dwarfs.

4.2. The non-relativistic $n = 3/2$ white dwarf

Equation (26) has

$$1 + \frac{1}{n} = \frac{5}{3}, \quad n = \frac{3}{2}. \quad (60)$$

For $n = 3/2$,

$$a^2 = \frac{5K_{\text{NR}}}{8\pi G} \rho_c^{-1/3}. \quad (61)$$

The mass and radius are

$$R = \xi_1 \left(\frac{5K_{\text{NR}}}{8\pi G} \right)^{1/2} \rho_c^{-1/6}, \quad (62)$$

and

$$M = 4\pi \left(\frac{5K_{\text{NR}}}{8\pi G} \right)^{3/2} \rho_c^{1/2} \xi_1^2 |\theta'(\xi_1)|. \quad (63)$$

Eliminating ρ_c gives

$$R = C_{3/2} \frac{K_{\text{NR}}}{G} M^{-1/3}, \quad (64)$$

where

$$C_{3/2} = \xi_1 \left[4\pi \xi_1^2 |\theta'(\xi_1)| \right]^{1/3} \frac{5}{8\pi}. \quad (65)$$

Thus

$$R \propto K_{\text{NR}} M^{-1/3} \propto \mu_e^{-5/3} M^{-1/3}. \quad (66)$$

A commonly used numerical form is

$$\begin{aligned} R_{\text{NR}} &\simeq 0.0112 R_{\odot} \left(\frac{\mu_e}{2} \right)^{-5/3} \\ &\times \left(\frac{M}{M_{\odot}} \right)^{-1/3}. \end{aligned} \quad (67)$$

Remark 4. The inverse mass-radius relation is the opposite of ordinary stars. Adding mass compresses the star, which increases the electron momenta and pressure. In the non-relativistic regime that pressure rise is strong enough to restore equilibrium at a smaller radius.

4.3. The ultra-relativistic $n = 3$ white dwarf

Equation (32) has

$$1 + \frac{1}{n} = \frac{4}{3}, \quad n = 3. \quad (68)$$

For $n = 3$,

$$a^2 = \frac{K_{\text{UR}}}{\pi G} \rho_c^{-2/3}. \quad (69)$$

The radius is

$$R = \xi_1 \left(\frac{K_{\text{UR}}}{\pi G} \right)^{1/2} \rho_c^{-1/3}, \quad (70)$$

but the mass is

$$\begin{aligned} M &= 4\pi \left(\frac{K_{\text{UR}}}{\pi G} \right)^{3/2} \xi_1^2 |\theta'(\xi_1)| \\ &\equiv M_{\text{Ch}}. \end{aligned} \quad (71)$$

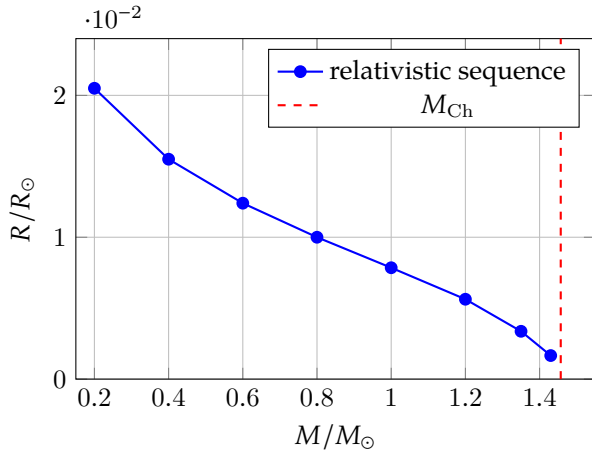


Figure 4. Approximate white-dwarf mass–radius relation for $\mu_e = 2$. Relativistic degeneracy drives the radius toward zero as the mass approaches the Chandrasekhar limit.

All dependence on ρ_c has cancelled. The $n = 3$ polytrope therefore represents a one-parameter family of configurations with the same mass and different radii. Increasing central density makes the radius smaller but does not raise the supported mass.

The numerical Lane–Emden constant for $n = 3$ is

$$\xi_1^2 |\theta'(\xi_1)| = 2.01824, \quad \xi_1 = 6.89685. \quad (72)$$

Substituting (33) into (71) gives

$$M_{\text{Ch}} = 4\pi \left[\frac{1}{\pi G} \frac{\hbar c}{4} (3\pi^2)^{1/3} (\mu_e m_u)^{-4/3} \right]^{3/2} \times \xi_1^2 |\theta'(\xi_1)|. \quad (73)$$

Collecting powers,

$$M_{\text{Ch}} = \frac{4}{\sqrt{\pi}} \xi_1^2 |\theta'(\xi_1)| \left(\frac{\hbar c}{4G} \right)^{3/2} \times (3\pi^2)^{1/2} \frac{1}{\mu_e^2 m_u^2}. \quad (74)$$

Evaluation with the constants in Table 1 yields

$$M_{\text{Ch}} = 5.83 \mu_e^{-2} M_{\odot}. \quad (75)$$

For carbon–oxygen matter,

$$\mu_e = 2, \quad M_{\text{Ch}} = 1.457 M_{\odot}. \quad (76)$$

Proposition 1 (Chandrasekhar mass in the ideal model). *A cold, Newtonian, non-rotating white dwarf supported by an ideal, completely degenerate electron gas has an upper mass*

$$M_{\text{max}} = M_{\text{Ch}} = 5.83 \mu_e^{-2} M_{\odot}. \quad (77)$$

For $\mu_e = 2$, this is $1.457 M_{\odot}$.

4.4. Stability and the role of the adiabatic index

The mass limit can also be interpreted through dynamical stability. For a sequence of equilibrium models with fixed composition and entropy, the onset of radial instability occurs at a turning point,

$$\frac{dM}{d\rho_c} = 0. \quad (78)$$

The non-relativistic sequence has $M \propto \rho_c^{1/2}$ and is therefore stable against this criterion, whereas the limiting $n = 3$ sequence has $dM/d\rho_c = 0$ and is marginal.

A related local statement follows from the virial theorem. A self-gravitating fluid with pressure response

$$\delta P = \Gamma_1 P \frac{\delta \rho}{\rho} \quad (79)$$

is stabilized by pressure if the volume-averaged adiabatic exponent exceeds

$$\Gamma_1 > \frac{4}{3}. \quad (80)$$

The exact electron gas has $\gamma(x) > 4/3$ for any finite x , but $\gamma(x)$ approaches $4/3$ from above as x grows. This explains why the Newtonian ideal sequence approaches marginality asymptotically rather than reaching a sharp instability at a finite central density. In real stars, general relativity, inverse beta decay, and Coulomb effects supply additional terms that move the instability to a finite density.

5. Heuristic Energy Derivation

The formal Lane–Emden calculation can be understood from a scaling argument. Let N be the number of baryons and $N_e \simeq N/\mu_e$ the number of electrons. If the star has radius R , then

$$n_e \sim \frac{N_e}{R^3}. \quad (81)$$

The Fermi momentum scales as

$$p_F \sim \hbar n_e^{1/3} \sim \frac{\hbar N_e^{1/3}}{R}. \quad (82)$$

In the ultra-relativistic regime, the electron kinetic energy is

$$E_{\text{kin}} \sim N_e p_F c \sim \frac{\hbar c N_e^{4/3}}{R}. \quad (83)$$

The gravitational energy is

$$E_{\text{grav}} \sim -\frac{GM^2}{R}. \quad (84)$$

Thus

$$E(R) \sim \frac{N^{4/3} \hbar c}{\mu_e^{4/3} R} - \frac{GM^2}{R}. \quad (85)$$

Since $M \simeq N m_u$, this may be written schematically as

$$E(R) \sim \frac{N^{4/3} \hbar c - GM^2}{R}, \quad (86)$$

where the composition factor and constants of order unity have been suppressed.

The crucial feature is that both ultra-relativistic degeneracy energy and gravitational energy scale as R^{-1} . If the coefficient of $1/R$ is positive, expansion is favored; if it is negative, collapse is favored; if it is zero, the configuration is marginal. The marginal condition is

$$\hbar c N^{4/3} \sim G m_u^2 N^2, \quad (87)$$

which implies

$$N \sim \left(\frac{\hbar c}{G m_u^2} \right)^{3/2}. \quad (88)$$

Therefore

$$M \sim m_u \left(\frac{\hbar c}{G m_u^2} \right)^{3/2} = \left(\frac{\hbar c}{G} \right)^{3/2} m_u^{-2}. \quad (89)$$

This is the Chandrasekhar scale. The exact coefficient, obtained by solving the $n = 3$ Lane–Emden problem, is the $5.83 \mu_e^{-2}$ in (75).

6. Worked Numerical Applications

6.1. Fermi momentum and pressure at a typical white-dwarf density

Consider carbon–oxygen matter with

$$\mu_e = 2, \quad \rho = 10^6 \text{ g cm}^{-3}. \quad (90)$$

Equation (8) gives

$$\begin{aligned} n_e &= \frac{10^6}{2(1.6605391 \times 10^{-24})} \text{ cm}^{-3} \\ &= 3.01 \times 10^{29} \text{ cm}^{-3}. \end{aligned} \quad (91)$$

The Fermi momentum is

$$\begin{aligned} p_F &= \hbar(3\pi^2 n_e)^{1/3} \\ &= 2.18 \times 10^{-17} \text{ g cm s}^{-1}. \end{aligned} \quad (92)$$

The dimensionless relativity parameter is

$$x = \frac{p_F}{m_e c} = 0.798. \quad (93)$$

Thus the electrons are mildly relativistic. The non-relativistic pressure estimate from (25) is

$$P_{\text{NR}} = 3.16 \times 10^{22} \text{ dyn cm}^{-2}, \quad (94)$$

whereas the full Chandrasekhar equation of state gives approximately

$$P \simeq 2.6 \times 10^{22} \text{ dyn cm}^{-2}. \quad (95)$$

The difference is already significant because x is not small. At higher densities the non-relativistic expression increasingly overestimates the stiffness of the gas.

6.2. Mass–radius estimate and Sirius B

Sirius B has a dynamical mass close to

$$M_{\text{Sirius B}} \simeq 1.02 M_{\odot}. \quad (96)$$

Using the non-relativistic mass–radius law (67) with $\mu_e = 2$ gives

$$\begin{aligned} R_{\text{NR}} &\simeq 0.0112 R_{\odot} (1.02)^{-1/3} \\ &\simeq 0.0111 R_{\odot}. \end{aligned} \quad (97)$$

In centimeters,

$$R_{\text{NR}} \simeq 7.7 \times 10^8 \text{ cm}. \quad (98)$$

A relativistic correction is necessary for such a massive white dwarf. An accurate analytic interpolation often used for illustration is [15]

$$\begin{aligned} R &\simeq 0.0112 R_{\odot} \left[\left(\frac{M_{\text{Ch}}}{M} \right)^{2/3} \right. \\ &\quad \left. - \left(\frac{M}{M_{\text{Ch}}} \right)^{2/3} \right]^{1/2}, \end{aligned} \quad (99)$$

with $M_{\text{Ch}} = 1.457 M_{\odot}$ for $\mu_e = 2$. For Sirius B,

$$\begin{aligned} R &\simeq 0.0112 R_{\odot} \left[\left(\frac{1.457}{1.02} \right)^{2/3} \right. \\ &\quad \left. - \left(\frac{1.02}{1.457} \right)^{2/3} \right]^{1/2} \\ &\simeq 0.0076 R_{\odot}. \end{aligned} \quad (100)$$

This is of the same order as the observed radius, about $8 \times 10^{-3} R_{\odot}$, and demonstrates why Sirius B was historically decisive: stellar mass confined to a radius comparable to Earth's requires degeneracy pressure, not ideal-gas pressure.

6.3. A massive carbon–oxygen white dwarf

As a second example, consider an ideal carbon–oxygen white dwarf of mass

$$M = 1.30 M_{\odot}, \quad \mu_e = 2. \quad (101)$$

Using (99) gives

$$\begin{aligned} R &\simeq 0.0112 R_{\odot} \left[\left(\frac{1.457}{1.30} \right)^{2/3} \right. \\ &\quad \left. - \left(\frac{1.30}{1.457} \right)^{2/3} \right]^{1/2} \\ &= 0.00437 R_{\odot}. \end{aligned} \quad (102)$$

Thus

$$R = 3.04 \times 10^8 \text{ cm}. \quad (103)$$

The mean density is

$$\bar{\rho} = \frac{3M}{4\pi R^3}$$

$$= 2.20 \times 10^7 \text{ g cm}^{-3}. \quad (104)$$

If the inner structure is approximated by the centrally condensed $n = 3$ polytrope, (59) gives

$$\rho_c \simeq 54.2 \bar{\rho} = 1.19 \times 10^9 \text{ g cm}^{-3}. \quad (105)$$

The central relativity parameter is then

$$x_c = \left(\frac{\rho_c}{\rho_0 \mu_e} \right)^{1/3} = 8.48. \quad (106)$$

The corresponding kinetic Fermi energy is

$$\begin{aligned} E_{F,\text{kin}} &= m_e c^2 \left[(1 + x_c^2)^{1/2} - 1 \right] \\ &= 3.87 \text{ MeV}. \end{aligned} \quad (107)$$

This value is large compared with thermal energies in an old white dwarf and illustrates why high-mass white dwarfs are deeply degenerate and relativistic in their cores.

The same model predicts an enormous surface gravity,

$$\begin{aligned} g &= \frac{GM}{R^2} \\ &= 1.87 \times 10^9 \text{ cm s}^{-2}. \end{aligned} \quad (108)$$

Its weak-field gravitational redshift, expressed as an equivalent velocity, is

$$\begin{aligned} v_{\text{gr}} &= zc \simeq \frac{GM}{Rc} \\ &= 1.89 \times 10^7 \text{ cm s}^{-1} \\ &= 189 \text{ km s}^{-1}. \end{aligned} \quad (109)$$

Such redshifts are measurable spectroscopically and provide an observational route to testing the mass-radius relation.

6.4. The Chandrasekhar mass for different compositions

For helium, carbon, and oxygen matter, $\mu_e \simeq 2$, and

$$M_{\text{Ch}} = 1.457 M_{\odot}. \quad (110)$$

For pure iron,

$$\mu_e = \frac{56}{26} = 2.154, \quad (111)$$

so

$$\begin{aligned} M_{\text{Ch,Fe}} &= 5.83(2.154)^{-2} M_{\odot} \\ &= 1.26 M_{\odot}. \end{aligned} \quad (112)$$

This illustrates that the limiting mass is controlled by the electron fraction. More neutron-rich material has fewer electrons per baryon and therefore less degeneracy pressure at fixed mass density.

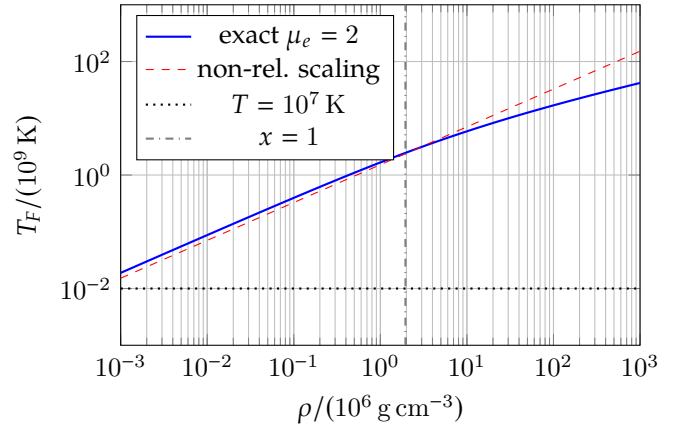


Figure 5. Fermi temperature as a function of density for $\mu_e = 2$. Even at modest white-dwarf densities, T_F is far above typical interior temperatures, validating the zero-temperature approximation for the electron pressure.

7. Extensions, Generalisations, and Observational Tests

7.1. Finite temperature and the degeneracy scale

White dwarfs are not at zero temperature, but their electrons are usually close to the zero-temperature limit. The kinetic Fermi energy is

$$E_{F,\text{kin}} = m_e c^2 \left[(1 + x^2)^{1/2} - 1 \right]. \quad (113)$$

The associated Fermi temperature is

$$T_F = \frac{E_{F,\text{kin}}}{k_B}. \quad (114)$$

Using (11), one obtains

$$x = \left(\frac{\rho}{\rho_0 \mu_e} \right)^{1/3}. \quad (115)$$

For $\mu_e = 2$ and $\rho = 10^6 \text{ g cm}^{-3}$,

$$T_F = 1.62 \times 10^9 \text{ K}. \quad (116)$$

Observed effective temperatures are typically 10^4 – 10^5 K , and even interior temperatures of young hot white dwarfs are usually far below T_F . The small parameter for thermal corrections is

$$\eta_T = \frac{k_B T}{E_{F,\text{kin}}} = \frac{T}{T_F}. \quad (117)$$

When $\eta_T \ll 1$, the leading thermal correction to a degenerate Fermi gas is of order

$$\frac{\Delta P}{P} \sim \eta_T^2. \quad (118)$$

Thermal pressure therefore affects the outer envelope and cooling luminosity much more strongly than it affects the global mass limit.

7.2. How the mass–radius relation is tested

A white-dwarf radius can be inferred from flux, distance, and effective temperature. If F_{bol} is the measured bolometric flux and D is the distance, then

$$L = 4\pi D^2 F_{\text{bol}} = 4\pi R^2 \sigma_{\text{SB}} T_{\text{eff}}^4. \quad (119)$$

Thus

$$R = D \left(\frac{F_{\text{bol}}}{\sigma_{\text{SB}} T_{\text{eff}}^4} \right)^{1/2}. \quad (120)$$

The mass can be obtained from binary dynamics, astrometry, or gravitational redshift. For a single-lined binary with orbital period P_{orb} and radial-velocity semi-amplitude K_1 , the mass function is

$$f(M) = \frac{P_{\text{orb}} K_1^3}{2\pi G} = \frac{(M_2 \sin i)^3}{(M_1 + M_2)^2}. \quad (121)$$

In favorable eclipsing systems the inclination i is known accurately, giving a dynamical mass.

Gravitational redshift provides an independent mass-to-radius measurement. In the weak-field limit,

$$z = \frac{\Delta\lambda}{\lambda} \simeq \frac{GM}{Rc^2}. \quad (122)$$

Equivalently, the redshift velocity is

$$v_{\text{gr}} = zc \simeq 0.636 \frac{M/M_{\odot}}{R/R_{\odot}} \text{ km s}^{-1}. \quad (123)$$

Combining (120) and (122) tests whether observed white dwarfs follow the predicted inverse mass–radius trend. The strongest tests use systems where atmosphere modeling, parallax, and dynamical masses are all available.

7.3. Beyond a single cold composition

The ideal model assumes a single value of μ_e . Real white dwarfs are layered, with a carbon–oxygen or oxygen–neon core, a helium mantle, and often a thin hydrogen atmosphere. A useful local generalization is

$$n_e(r) = \frac{\rho(r)}{m_u} \sum_i \frac{Z_i X_i(r)}{A_i}. \quad (124)$$

The hydrostatic equations then remain (46) and (47), but the equation of state becomes position dependent through the composition profile. If the composition changes only in a low-mass outer layer, the effect on M_{Ch} is small because most of the mass lies in the core. If the core becomes neutron-rich, however, the decrease in electron fraction directly lowers the limiting mass.

A convenient measure of the electron fraction is

$$Y_e = \frac{1}{\mu_e}. \quad (125)$$

The ideal limit may be written

$$M_{\text{Ch}} = 5.83 Y_e^2 M_{\odot}. \quad (126)$$

This form makes clear why electron captures destabilize dense cores: they reduce Y_e , and the mass that can be supported by degeneracy pressure falls as Y_e^2 .

8. Discussion

8.1. Assumptions and domain of validity

The ideal Chandrasekhar derivation rests on several assumptions. First, the electrons are taken to be completely degenerate. This is excellent for mature white dwarfs because their Fermi energies are typically much larger than $k_B T$. Second, the ions provide mass but negligible pressure. This is usually valid dynamically, although ionic thermal energy controls cooling and crystallization. Third, Newtonian gravity is used. For most white dwarfs $GM/(Rc^2) \ll 1$, but relativistic gravity becomes relevant near the limiting mass. Fourth, the electron gas is ideal, with no Coulomb, exchange, magnetic, or lattice corrections. These terms are small compared with the leading degeneracy pressure but important for precision mass–radius calculations.

The exact equation of state (37) also assumes uniform local composition and zero temperature. In real stars, the composition may be stratified: a carbon–oxygen core, helium layer, and hydrogen atmosphere are common. Such layering modifies observed radii and cooling ages but does not change the leading Chandrasekhar scale because the core has $\mu_e \simeq 2$.

8.2. Inverse beta decay and neutronization

At sufficiently high density, electron Fermi energies become large enough to trigger electron captures,

$$e^- + (A, Z) \rightarrow (A, Z - 1) + \nu_e. \quad (127)$$

The threshold condition is approximately

$$E_F \gtrsim Q_{\text{nuc}}, \quad (128)$$

where Q_{nuc} is the relevant nuclear mass-energy difference. Electron capture reduces n_e at fixed baryon density and hence lowers degeneracy pressure. In oxygen–neon–magnesium cores this process can induce collapse before the ideal Chandrasekhar mass is reached. In carbon–oxygen white dwarfs, thermonuclear runaway generally intervenes before extensive neutronization in the progenitor core.

8.3. Coulomb and lattice corrections

The ions form a strongly coupled plasma when their Coulomb energy exceeds their thermal energy. The coupling parameter is

$$\Gamma = \frac{Z^2 e^2}{a_i k_B T}, \quad (129)$$

where

$$a_i = \left(\frac{3}{4\pi n_i} \right)^{1/3} \quad (130)$$

is the ion-sphere radius. Coulomb interactions lower the total energy and slightly reduce the pressure compared with the ideal electron gas. At low temperature the ions crystallize, affecting white-dwarf cooling

and pulsations. The Hamada–Salpeter mass–radius relations [8] include such corrections and are more accurate than the ideal formula for detailed comparison with observations. Salpeter [9] gives the corresponding zero-temperature plasma corrections.

8.4. General relativity

General relativity strengthens gravity relative to Newtonian theory [12, 13]. Hydrostatic equilibrium is then governed by the Tolman–Oppenheimer–Volkoff equation,

$$\frac{dP}{dr} = -\frac{G}{r^2} \frac{[\epsilon + P/c^2] [m + 4\pi r^3 P/c^2]}{1 - 2Gm/(rc^2)}, \quad (131)$$

with

$$\frac{dm}{dr} = 4\pi r^2 \epsilon. \quad (132)$$

Here ϵ is mass-energy density. For ordinary white dwarfs the compactness is small,

$$\frac{GM}{Rc^2} \sim 10^{-4} - 10^{-3}, \quad (133)$$

so Newtonian theory is usually accurate. Near the maximum mass, however, general relativity lowers the stable mass slightly and contributes to the instability of the highest-density configurations.

8.5. Rotation and magnetic fields

Uniform rotation increases the maximum mass because centrifugal support reduces the effective gravity. A rough measure is the ratio

$$\frac{T}{|W|} \sim \frac{\Omega^2 R^3}{GM}, \quad (134)$$

where T is rotational kinetic energy and W gravitational binding energy. Rotation becomes dynamically important near the Keplerian angular velocity

$$\Omega_K \simeq \left(\frac{GM}{R^3} \right)^{1/2}. \quad (135)$$

Differential rotation can support larger masses temporarily, but magnetic torques and viscosity tend to redistribute angular momentum. Strong magnetic fields can modify electron phase space through Landau quantization, but fields large enough to alter the global Chandrasekhar limit are extreme and must be treated with magnetic stresses and stability constraints.

8.6. Observational white-dwarf masses

Observed white-dwarf masses cluster near

$$M_{\text{peak}} \simeq 0.6 M_{\odot}, \quad (136)$$

well below M_{Ch} . This is not a contradiction. Most white dwarfs are the exposed cores of low- and intermediate-mass stars whose envelopes were lost on the red-giant

and asymptotic-giant branches. Stellar evolution, not the Chandrasekhar limit, determines the typical remnant mass. The limit instead determines the upper end of the stable white-dwarf distribution and the fate of accreting or merging systems.

In close binaries, a carbon–oxygen white dwarf may approach the limiting mass by accretion or merge with another white dwarf. Compressional heating, carbon ignition, convection, electron captures, and angular momentum transport then determine whether the outcome is a Type Ia supernova, accretion-induced collapse, or a bound remnant. The Chandrasekhar mass remains central because it sets the natural mass scale at which degeneracy-supported carbon–oxygen matter becomes globally unstable to thermonuclear disruption.

9. Conclusion

The Chandrasekhar limit follows from a chain of elementary but profound arguments. Pauli exclusion fixes the density of electron momentum states and yields

$$p_F = \hbar(3\pi^2 n_e)^{1/3}. \quad (137)$$

The pressure is the isotropic momentum flux,

$$P = \frac{1}{3} \int_0^{p_F} p v(p) n(p) dp, \quad (138)$$

which gives $P \propto n_e^{5/3}$ for non-relativistic electrons and $P \propto n_e^{4/3}$ for ultra-relativistic electrons. The exact Chandrasekhar equation of state smoothly connects these limits through the function $f(x)$ in (39).

When inserted into hydrostatic equilibrium, the non-relativistic equation of state produces an $n = 3/2$ polytrope and the inverse mass–radius law $R \propto M^{-1/3}$. The ultra-relativistic equation of state produces an $n = 3$ polytrope whose mass is independent of radius. Solving the Lane–Emden problem gives the ideal limiting mass

$$M_{\text{Ch}} = 5.83 \mu_e^{-2} M_{\odot}, \quad (139)$$

or $1.457 M_{\odot}$ for $\mu_e = 2$. This result is one of the great quantitative successes of applying quantum mechanics and relativity to astrophysics. Its ideal assumptions are modified in real white dwarfs by Coulomb physics, inverse beta decay, finite temperature, rotation, magnetic fields, and general relativity, but the fundamental mass scale remains the same. The observed concentration of white dwarfs near $0.6 M_{\odot}$ reflects stellar evolution, while the Chandrasekhar mass governs the high-mass endpoint and the physics of Type Ia supernova progenitors.

A. Detailed Algebra for the Pressure Integral

This appendix records the steps leading to the closed form in (36). Define

$$I(x) = \int_0^x \frac{y^4 dy}{(1+y^2)^{1/2}}. \quad (140)$$

Use the substitution

$$y = \sinh t, \quad dy = \cosh t \, dt. \quad (141)$$

Then $(1+y^2)^{1/2} = \cosh t$, and

$$I(x) = \int_0^{\sinh^{-1} x} \sinh^4 t \, dt. \quad (142)$$

The identity

$$\sinh^4 t = \frac{1}{8} \cosh 4t - \frac{1}{2} \cosh 2t + \frac{3}{8} \quad (143)$$

gives

$$I(x) = \left[\frac{1}{32} \sinh 4t - \frac{1}{4} \sinh 2t + \frac{3}{8} t \right]_0^{\sinh^{-1} x}. \quad (144)$$

With

$$\sinh 2t = 2x(1+x^2)^{1/2}, \quad (145)$$

and

$$\sinh 4t = 4x(1+2x^2)(1+x^2)^{1/2}, \quad (146)$$

equation (144) becomes

$$I(x) = \frac{1}{8} x(2x^2 - 3)(1+x^2)^{1/2} + \frac{3}{8} \sinh^{-1} x, \quad (147)$$

which is (36).

The derivative of the Chandrasekhar function is especially simple:

$$f'(x) = 8 \frac{x^4}{(1+x^2)^{1/2}}. \quad (148)$$

This follows either by differentiating (39) directly or by differentiating (140). It is the identity used in the stiffness formula (42).

B. Series Solution of the Lane–Emden Equation Near the Center

Numerical integrations of the Lane–Emden equation start at small but nonzero ξ , because (51) contains the apparently singular factor $1/\xi^2$. Regularity at the center removes the singularity. Write

$$\theta(\xi) = 1 + a_2 \xi^2 + a_4 \xi^4 + O(\xi^6). \quad (149)$$

Then

$$\theta'(\xi) = 2a_2 \xi + 4a_4 \xi^3 + O(\xi^5), \quad (150)$$

and

$$\frac{1}{\xi^2} \frac{d}{d\xi} (\xi^2 \theta') = 6a_2 + 20a_4 \xi^2 + O(\xi^4). \quad (151)$$

The right-hand side is

$$-\theta^n = -1 - na_2 \xi^2 + O(\xi^4). \quad (152)$$

Equating coefficients gives

$$a_2 = -\frac{1}{6}, \quad a_4 = \frac{n}{120}. \quad (153)$$

Thus the regular central expansion is

$$\theta(\xi) = 1 - \frac{\xi^2}{6} + \frac{n\xi^4}{120} + O(\xi^6). \quad (154)$$

This expansion supplies stable starting values for computing ξ_1 and $\theta'(\xi_1)$.

The mass identity (56) follows by multiplying (51) by ξ^2 :

$$\frac{d}{d\xi} (\xi^2 \theta') = -\xi^2 \theta^n. \quad (155)$$

Integrating from the center to the surface gives

$$[\xi^2 \theta']_0^{\xi_1} = - \int_0^{\xi_1} \xi^2 \theta^n \, d\xi. \quad (156)$$

The central boundary condition makes the lower limit vanish, so

$$\int_0^{\xi_1} \xi^2 \theta^n \, d\xi = -\xi_1^2 \theta'(\xi_1). \quad (157)$$

This is the relation that converts the density integral into the compact mass formula (57).

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