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EXTREME PHYSICS

# **Derivation of the Tolman-Oppenheimer-Volkoff (TOV) Equilibrium Equation in General Relativity and Neutron Star Structure**

Academic Research Paper

Theoretical Astrophysics — Step-by-Step Analytic Derivations, Vector Diagrams and Numerical Tables

# Derivation of the Tolman–Oppenheimer–Volkoff (TOV) Equilibrium Equation in General Relativity and Neutron Star Structure

Extreme Physics

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**Abstract.** The Tolman–Oppenheimer–Volkoff equation is the relativistic equation of hydrostatic equilibrium for a static, spherical perfect-fluid star. It replaces the Newtonian balance between pressure gradients and gravitational force by a geometrical balance in curved spacetime, where energy density, pressure, and spacetime curvature are inseparable. This paper gives a self-contained derivation from the static spherically symmetric metric, the perfect-fluid stress-energy tensor, the Einstein field equations, and local stress-energy conservation. The mass equation, gravitational-potential equation, and pressure equation are obtained explicitly from the  $tt$  and  $rr$  components of the Einstein tensor and from  $\nabla_\mu T^{\mu r} = 0$ . Their combination yields the TOV equation in a factored form displaying its three relativistic corrections: pressure inertia, pressure self-gravity, and curvature amplification. We also derive the uniform-density Schwarzschild interior solution, the divergence of its central pressure, and the Buchdahl bound. Additional sections give dimensionless forms, numerical estimates, observational diagnostics, and algebraic details useful for implementation. Applications to neutron-star structure, equations of state, compactness, gravitational redshift, numerical integration, binding energy, and observed two-solar-mass pulsars are discussed.

**Keywords:** Tolman–Oppenheimer–Volkoff equation; neutron stars; general relativity; hydrostatic equilibrium; compactness; equation of state

## 1. Introduction

The problem of stellar equilibrium is one of the central organizing problems of astrophysics. In Newtonian theory, a static spherical star is described by two elementary relations: the enclosed mass  $m(r)$  grows according to the local mass density, and the pressure gradient balances the gravitational force per unit volume. These equations are sufficient for ordinary stars when supplemented by energy transport and an equation of state [1, 10]. Their conceptual simplicity conceals a profound limitation: gravity is treated as an external force acting on matter, while pressure contributes only to mechanical support. For white dwarfs, neutron stars, and hypothetical ultra-compact objects, this separation fails. The gravitational field is no longer weak, internal velocities may be relativistic, and the internal pressure is comparable to the rest-mass energy density. Hydrostatic equilibrium must then be derived from general relativity.

Historically, the relativistic stellar-structure equation was developed shortly after the Schwarzschild exterior solution. The interior constant-density solution was found by Schwarzschild in 1916 [2]; Tolman systematized relativistic fluid spheres [3]; and Oppenheimer and Volkoff applied the theory to cold neutron matter, obtaining the first relativistic neutron-star models [4]. The resulting equation is now called the Tolman–Oppenheimer–Volkoff (TOV) equation. It is a cornerstone of compact-object astrophysics, nuclear astrophysics, and strong-field gravity [6, 7, 8].

Neutron stars provide the most important physical realization of TOV equilibrium. A neutron star of mass

$M \simeq 1.4 M_\odot$  and radius  $R \simeq 10\text{--}14\text{ km}$  has compactness  $GM/(Rc^2) \sim 0.15\text{--}0.25$ , surface gravity exceeding  $10^{12}\text{ m s}^{-2}$ , and mean density of order nuclear saturation density. In this regime, the pressure may be a substantial fraction of the energy density, and relativistic corrections are not perturbative details. They determine whether an equation of state can support a two-solar-mass star and whether collapse to a black hole is unavoidable.

The purpose of this paper is to derive the TOV equation in a complete but pedagogical form appropriate for advanced undergraduate and graduate astrophysics. The derivation begins with the static spherical metric

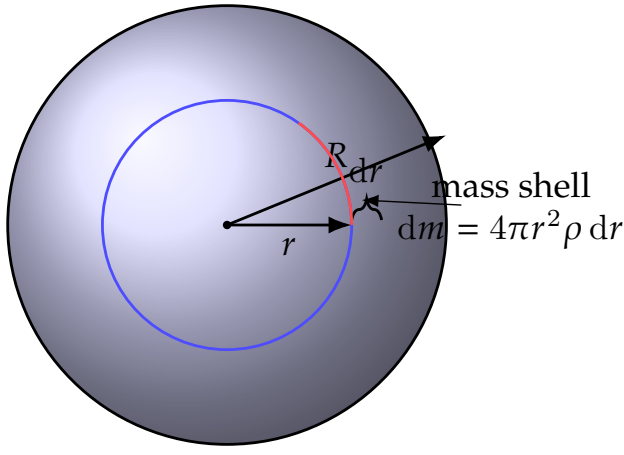
$$ds^2 = -e^{2\Phi(r)}c^2 dt^2 + \left(1 - \frac{2Gm(r)}{rc^2}\right)^{-1} dr^2 + r^2 d\Omega^2, \quad (1)$$

where

$$d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2. \quad (2)$$

The function  $m(r)$  will become the gravitational mass enclosed within areal radius  $r$ , while  $\Phi(r)$  is the logarithmic gravitational redshift potential.

A key message is that the TOV equation is not obtained by adding relativistic terms by hand. It follows from two principles: Einstein’s equations determine curvature from the stress-energy tensor, and local stress-energy conservation determines force balance. In the Newtonian limit the equation reduces smoothly to the familiar equation of hydrostatic equilibrium, but in the relativistic regime it contains three correction factors with clear physical meanings.



$$g_{tt} = -e^{2\Phi(r)}c^2$$

$$g_{rr} = \left(1 - 2Gm(r)/(rc^2)\right)^{-1}$$

**Figure 1.** Schematic of a static spherical relativistic star. The coordinate  $r$  is the areal radius, so a sphere has area  $4\pi r^2$ . The metric functions  $\Phi(r)$  and  $m(r)$  encode gravitational redshift and enclosed gravitational mass.

## 2. Physical and Mathematical Framework

### 2.1. Static spherical geometry

We assume a spacetime that is time independent, spherically symmetric, and free of rotation, magnetic stresses, anisotropic stresses, and heat flow. The most general static spherical line element may first be written as

$$ds^2 = -e^{2\Phi(r)}c^2dt^2 + e^{2\Lambda(r)}dr^2 + r^2d\Omega^2. \quad (3)$$

The coordinate  $r$  is chosen as the areal radius:

$$A(r) = 4\pi r^2. \quad (4)$$

It is useful to define the mass function by

$$e^{-2\Lambda(r)} = 1 - \frac{2Gm(r)}{rc^2}. \quad (5)$$

Substitution of (5) into (3) gives (1). The quantity

$$C(r) = \frac{Gm(r)}{rc^2} \quad (6)$$

is the local compactness. When  $C \ll 1$ , spacetime curvature is weak and Newtonian gravity is recovered.

**Definition 1.** For a static spherical star, the stellar surface  $R$  is defined by

$$P(R) = 0, \quad (7)$$

and the gravitational mass measured by an observer at infinity is

$$M = m(R). \quad (8)$$

Regularity at the origin imposes additional conditions that are sometimes left implicit. Because the center is not a boundary surface but a locally flat point, the enclosed mass must vanish faster than  $r$ :

$$m(0) = 0, \quad \lim_{r \rightarrow 0} \frac{m(r)}{r} = 0. \quad (9)$$

The metric coefficient  $g_{rr}$  must approach unity, giving

$$e^{2\Lambda(0)} = 1. \quad (10)$$

A finite central pressure and density require

$$P(0) = P_c < \infty, \quad \rho(0) = \rho_c < \infty. \quad (11)$$

Spherical symmetry also implies that scalar gradients vanish at the center:

$$P'(0) = 0, \quad \rho'(0) = 0. \quad (12)$$

These conditions are useful both analytically and numerically, because they identify the nonsingular branch of the differential equations.

Outside the star,  $P = \rho = 0$ , and the metric must match the Schwarzschild exterior:

$$ds^2 = -\left(1 - \frac{2GM}{rc^2}\right)c^2dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1}dr^2 + r^2d\Omega^2. \quad (13)$$

Continuity of  $g_{tt}$  at  $r = R$  implies

$$e^{2\Phi(R)} = 1 - \frac{2GM}{Rc^2}. \quad (14)$$

### 2.2. Perfect-fluid matter

The stress-energy tensor of a perfect fluid is

$$T^{\mu\nu} = \left(\rho + \frac{P}{c^2}\right)u^\mu u^\nu + P g^{\mu\nu}, \quad (15)$$

where  $\rho$  is the mass-energy density, so that the local energy density is  $\epsilon = \rho c^2$ ,  $P$  is the isotropic pressure, and  $u^\mu$  is the four-velocity. The normalization condition is

$$g_{\mu\nu}u^\mu u^\nu = -c^2. \quad (16)$$

For a static fluid in the coordinates of (1),

$$u^r = u^\theta = u^\phi = 0. \quad (17)$$

Using (1) and (16),

$$-e^{2\Phi}c^2(u^t)^2 = -c^2, \quad (18)$$

so

$$u^t = e^{-\Phi}. \quad (19)$$

The mixed components of the stress-energy tensor are therefore

$$T^\mu{}_\nu = \text{diag}(-\rho c^2, P, P, P). \quad (20)$$

*Remark 1.* Equation (20) is deceptively simple. Its first component says that energy density sources curvature; the spatial components say that pressure also sources curvature. This fact has no Newtonian analogue and is central to compact-star structure.

### 2.3. Einstein equations

We adopt the sign convention

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}. \quad (21)$$

Equivalently, for mixed components,

$$G^\mu{}_\nu = \frac{8\pi G}{c^4} T^\mu{}_\nu. \quad (22)$$

The two components required for spherical equilibrium are the  $tt$  and  $rr$  equations. They determine  $m'(r)$  and  $\Phi'(r)$ , respectively. The pressure gradient then follows from local conservation,

$$\nabla_\mu T^{\mu\nu} = 0. \quad (23)$$

### 3. The Mass Equation from the $tt$ Field Equation

For the metric (3), the mixed time component of the Einstein tensor is

$$G^t{}_t = -\frac{1}{r^2} + e^{-2\Lambda} \left( \frac{1}{r^2} - \frac{2\Lambda'}{r} \right). \quad (24)$$

Here and below a prime denotes  $d/dr$ . Using (5),

$$e^{-2\Lambda} = 1 - \frac{2Gm}{rc^2}. \quad (25)$$

Differentiating (25) gives

$$-2\Lambda' e^{-2\Lambda} = -\frac{2G}{c^2} \left( \frac{m'}{r} - \frac{m}{r^2} \right). \quad (26)$$

Thus

$$e^{-2\Lambda} \Lambda' = \frac{G}{c^2} \left( \frac{m'}{r} - \frac{m}{r^2} \right). \quad (27)$$

Substitution into (24) yields

$$\begin{aligned} G^t{}_t &= -\frac{1}{r^2} + \frac{1}{r^2} \left( 1 - \frac{2Gm}{rc^2} \right) - \frac{2}{r} e^{-2\Lambda} \Lambda' \\ &= -\frac{2Gm}{r^3 c^2} - \frac{2G}{c^2} \left( \frac{m'}{r^2} - \frac{m}{r^3} \right) \\ &= -\frac{2Gm'}{c^2 r^2}. \end{aligned} \quad (28)$$

The field equation (22), together with (20), gives

$$G^t{}_t = -\frac{8\pi G}{c^2} \rho. \quad (29)$$

Equating (28) and (29) gives the relativistic mass equation

$$\frac{dm}{dr} = 4\pi r^2 \rho. \quad (30)$$

Integrating from the center,

$$m(r) = 4\pi \int_0^r \rho(\tilde{r}) \tilde{r}^2 d\tilde{r}. \quad (31)$$

The proper radial distance differs from the coordinate increment:

$$d\ell = \left( 1 - \frac{2Gm}{rc^2} \right)^{-1/2} dr. \quad (32)$$

Consequently the proper volume of a thin shell is

$$dV_{\text{prop}} = 4\pi r^2 \left( 1 - \frac{2Gm}{rc^2} \right)^{-1/2} dr. \quad (33)$$

If  $\rho_0$  denotes rest-mass density, the baryonic rest mass inside  $r$  is therefore

$$m_b(r) = 4\pi \int_0^r \rho_0(\tilde{r}) \left( 1 - \frac{2Gm}{\tilde{r}c^2} \right)^{-1/2} \tilde{r}^2 d\tilde{r}. \quad (34)$$

The difference between  $m_b(R)$  and  $M$  is a coordinate-invariant way to quantify gravitational binding once the conserved baryon number has been specified.

*Remark 2.* Although (30) looks Newtonian, its interpretation is relativistic. The density  $\rho$  is local energy density divided by  $c^2$ , while  $m(r)$  is the gravitational mass appearing in the curvature component  $g_{rr}$ . Proper volume contains the factor  $(1 - 2Gm/rc^2)^{-1/2}$ , so  $m(r)$  is not simply the integral of density over proper volume; gravitational binding energy is already encoded geometrically.

### 4. The Potential Equation from the $rr$ Field Equation

The mixed radial component of the Einstein tensor for (3) is

$$G^r{}_r = -\frac{1}{r^2} + e^{-2\Lambda} \left( \frac{1}{r^2} + \frac{2\Phi'}{r} \right). \quad (35)$$

Using (5), this becomes

$$\begin{aligned} G^r{}_r &= -\frac{1}{r^2} + \left( 1 - \frac{2Gm}{rc^2} \right) \left( \frac{1}{r^2} + \frac{2\Phi'}{r} \right) \\ &= -\frac{2Gm}{r^3 c^2} + \frac{2\Phi'}{r} \left( 1 - \frac{2Gm}{rc^2} \right). \end{aligned} \quad (36)$$

The source term is

$$G^r{}_r = \frac{8\pi G}{c^4} P. \quad (37)$$

Equating (36) and (37),

$$-\frac{2Gm}{r^3 c^2} + \frac{2\Phi'}{r} \left( 1 - \frac{2Gm}{rc^2} \right) = \frac{8\pi GP}{c^4}. \quad (38)$$

Solving step by step,

$$\frac{2\Phi'}{r} \left( 1 - \frac{2Gm}{rc^2} \right) = \frac{2Gm}{r^3 c^2} + \frac{8\pi GP}{c^4}, \quad (39)$$

and hence

$$\Phi' = \frac{Gm}{r^2 c^2} \left( 1 - \frac{2Gm}{rc^2} \right)^{-1} + \frac{4\pi GPr}{c^4} \left( 1 - \frac{2Gm}{rc^2} \right)^{-1}. \quad (40)$$

Combining the two terms gives

$$\frac{d\Phi}{dr} = \frac{G(m + 4\pi r^3 P/c^2)}{c^2 r^2 (1 - 2Gm/(rc^2))}. \quad (41)$$

*Remark 3.* The numerator of (41) contains  $m + 4\pi r^3 P/c^2$ . Pressure therefore contributes directly to the gravitational potential gradient. In relativistic gravity, pressure is not merely support against gravity; it is also a source of gravity.

It is sometimes useful to introduce a potential with Newtonian units,

$$\varphi(r) = c^2 \Phi(r). \quad (42)$$

Then (41) becomes

$$\frac{d\varphi}{dr} = \frac{G(m + 4\pi r^3 P/c^2)}{r^2 (1 - 2Gm/(rc^2))}. \quad (43)$$

In the weak-field and low-pressure limit this reduces to

$$\frac{d\varphi}{dr} \simeq \frac{Gm}{r^2}, \quad (44)$$

the usual Newtonian gravitational acceleration.

## 5. Stress-Energy Conservation and Hydrostatic Balance

Einstein's equations imply local stress-energy conservation through the contracted Bianchi identity. Nevertheless, deriving the radial component explicitly is the clearest route to the hydrostatic equation. The conservation equation is

$$\nabla_\mu T^{\mu r} = \partial_\mu T^{\mu r} + \Gamma^\mu_{\mu\alpha} T^{\alpha r} + \Gamma^r_{\mu\alpha} T^{\mu\alpha} = 0. \quad (45)$$

For a static perfect fluid at rest, the only nonzero diagonal contravariant components are

$$T^{tt} = \rho e^{-2\Phi}, \quad T^{rr} = P g^{rr}, \quad T^{\theta\theta} = \frac{P}{r^2}, \quad T^{\phi\phi} = \frac{P}{r^2 \sin^2 \theta}. \quad (46)$$

The relevant Christoffel symbols are

$$\Gamma^t_{tr} = \Phi', \quad \Gamma^r_{tt} = c^2 e^{2\Phi-2\Lambda} \Phi', \quad (47)$$

$$\Gamma^r_{rr} = \Lambda', \quad \Gamma^\theta_{\theta r} = \Gamma^\phi_{\phi r} = \frac{1}{r}, \quad (48)$$

and

$$\Gamma^r_{\theta\theta} = -r e^{-2\Lambda}, \quad \Gamma^r_{\phi\phi} = -r \sin^2 \theta e^{-2\Lambda}. \quad (49)$$

Because  $T^{\mu r}$  vanishes except for  $\mu = r$ , the first two terms in (45) give

$$\partial_\mu T^{\mu r} + \Gamma^\mu_{\mu\alpha} T^{\alpha r} = \frac{d}{dr} (P e^{-2\Lambda}) + \left( \Phi' + \Lambda' + \frac{2}{r} \right) P e^{-2\Lambda}. \quad (50)$$

The last term gives

$$\Gamma^r_{\mu\alpha} T^{\mu\alpha} = \Gamma^r_{tt} T^{tt} + \Gamma^r_{rr} T^{rr} + \Gamma^r_{\theta\theta} T^{\theta\theta} + \Gamma^r_{\phi\phi} T^{\phi\phi}$$

$$= \rho c^2 e^{-2\Lambda} \Phi' + \Lambda' P e^{-2\Lambda} - \frac{2P}{r} e^{-2\Lambda}. \quad (51)$$

Expanding the radial derivative,

$$\frac{d}{dr} (P e^{-2\Lambda}) = e^{-2\Lambda} P' - 2\Lambda' P e^{-2\Lambda}. \quad (52)$$

Substituting (50)–(52) into (45), all  $\Lambda'$  and  $2P/r$  terms cancel, leaving

$$e^{-2\Lambda} [P' + (\rho c^2 + P)\Phi'] = 0. \quad (53)$$

Therefore

$$\frac{dP}{dr} = -(\rho c^2 + P) \frac{d\Phi}{dr}. \quad (54)$$

Using (42), this can also be written as

$$\frac{dP}{dr} = -\left(\rho + \frac{P}{c^2}\right) \frac{d\varphi}{dr}. \quad (55)$$

*Remark 4.* Equation (55) is the exact relativistic analogue of Newtonian hydrostatic balance. The inertial mass density resisting acceleration is not  $\rho$  but  $\rho + P/c^2$ . Pressure contributes to inertia as well as to gravity.

A compact way to express the same result is through the four-acceleration of a static fluid element,

$$a^\mu = u^\nu \nabla_\nu u^\mu. \quad (56)$$

For the metric (3), the radial component is

$$a^r = c^2 e^{-2\Lambda} \Phi'. \quad (57)$$

The Euler equation obtained by projecting  $\nabla_\mu T^{\mu\nu} = 0$  orthogonally to  $u^\mu$  is

$$(\rho c^2 + P) a_\alpha + \left( g_\alpha{}^\mu + \frac{u_\alpha u^\mu}{c^2} \right) \nabla_\mu P = 0. \quad (58)$$

Taking  $\alpha = r$  gives (54). This form makes clear that hydrostatic equilibrium is local force balance in the instantaneous rest frame of each fluid element.

## 6. The Tolman-Oppenheimer-Volkoff Equation

Substituting (41) into (54) gives

$$\frac{dP}{dr} = -(\rho c^2 + P) \frac{G(m + 4\pi r^3 P/c^2)}{c^2 r^2 (1 - 2Gm/(rc^2))}. \quad (59)$$

Equivalently,

$$\frac{dP}{dr} = -\frac{Gm\rho}{r^2} \left( 1 + \frac{P}{\rho c^2} \right) \left( 1 + \frac{4\pi r^3 P}{mc^2} \right) \left( 1 - \frac{2Gm}{rc^2} \right)^{-1}. \quad (60)$$

Together with

$$\frac{dm}{dr} = 4\pi r^2 \rho, \quad (61)$$

and an equation of state

$$P = P(\rho), \quad (62)$$

equation (60) determines the structure of a cold spherical relativistic star.

The three correction factors in (60) are

$$I = 1 + \frac{P}{\rho c^2}, \quad (63)$$

$$S = 1 + \frac{4\pi r^3 P}{mc^2}, \quad (64)$$

and

$$\mathcal{G} = \left(1 - \frac{2Gm}{rc^2}\right)^{-1}. \quad (65)$$

Thus

$$\frac{dP}{dr} = -\frac{Gm\rho}{r^2} I S \mathcal{G}. \quad (66)$$

The Newtonian limit is obtained by assuming

$$\frac{P}{\rho c^2} \ll 1, \quad \frac{4\pi r^3 P}{mc^2} \ll 1, \quad \frac{2Gm}{rc^2} \ll 1. \quad (67)$$

Then

$$I \simeq S \simeq \mathcal{G} \simeq 1, \quad (68)$$

and (60) becomes

$$\frac{dP}{dr} = -\frac{Gm\rho}{r^2}. \quad (69)$$

Together with (30), this is the ordinary equation of stellar hydrostatic equilibrium.

## 7. Dimensionless Form and Scaling

For numerical work it is useful to scale out the large dimensional constants. Choose a characteristic density  $\rho_*$  and define

$$r_* = \frac{c}{\sqrt{4\pi G \rho_*}}, \quad m_* = 4\pi \rho_* r_*^3. \quad (70)$$

Introduce dimensionless variables

$$x = \frac{r}{r_*}, \quad \mu = \frac{m}{m_*}, \quad p = \frac{P}{\rho_* c^2}, \quad e = \frac{\rho}{\rho_*}. \quad (71)$$

Then the mass equation becomes

$$\frac{d\mu}{dx} = x^2 e. \quad (72)$$

The compactness factor is

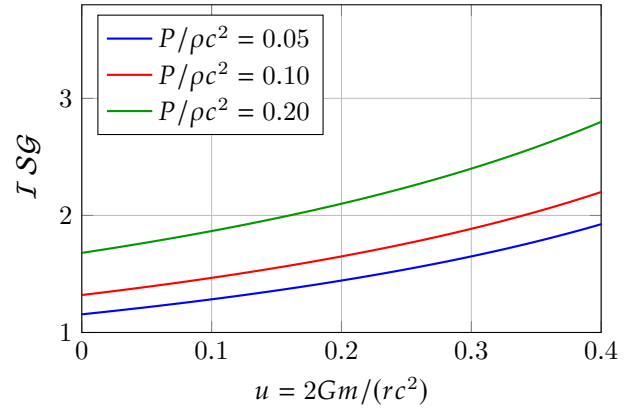
$$\frac{2Gm}{rc^2} = \frac{2\mu}{x}. \quad (73)$$

The TOV pressure equation becomes

$$\frac{dp}{dx} = -\frac{(e+p)(\mu + x^3 p)}{x^2(1 - 2\mu/x)}. \quad (74)$$

The potential equation is similarly

$$\frac{d\Phi}{dx} = \frac{\mu + x^3 p}{x^2(1 - 2\mu/x)}. \quad (75)$$



**Figure 2.** Combined TOV correction factor as a function of local compactness  $u = 2Gm/(rc^2)$  for three representative pressure levels. The plotted curves use  $4\pi r^3 P/(mc^2) = 2P/(\rho c^2)$  as a simple illustrative scaling.

Equations (72) and (74) show that all dimensional information enters through the equation of state and the chosen density scale. For example, if

$$\rho_* = 10^{18} \text{ kg m}^{-3}, \quad (76)$$

then

$$r_* = 10.36 \text{ km}, \quad m_* = 6.98 M_\odot. \quad (77)$$

A dimensionless mass  $\mu = 0.30$  therefore corresponds to  $M \approx 2.09 M_\odot$ .

Figure 2 illustrates why relativistic corrections become large before a horizon is approached. At  $u = 0.35$ , even moderate pressure ratios lead to a combined correction of order two. This behavior is typical of neutron-star interiors and explains why Newtonian stellar models overestimate the support provided by a given pressure profile.

## 8. Uniform-Density Schwarzschild Interior Solution

The simplest exact relativistic star has constant density:

$$\rho(r) = \rho_0 = \text{constant}. \quad (78)$$

The mass equation gives

$$m(r) = \frac{4\pi}{3} \rho_0 r^3. \quad (79)$$

At the surface,

$$M = \frac{4\pi}{3} \rho_0 R^3. \quad (80)$$

Define the compactness parameter

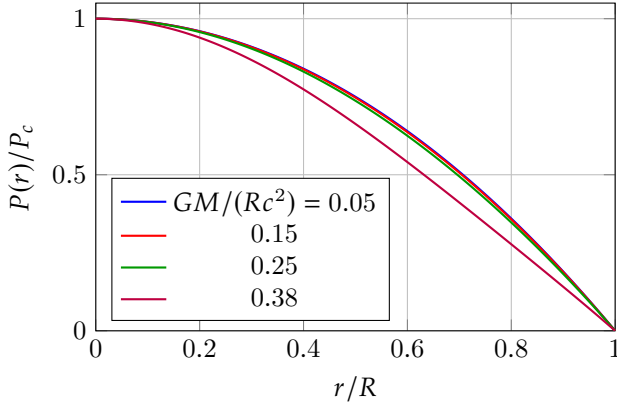
$$\beta = \frac{GM}{Rc^2}. \quad (81)$$

Then

$$\frac{2Gm(r)}{rc^2} = 2\beta \frac{r^2}{R^2}. \quad (82)$$

**Table 1.** Newtonian hydrostatic equilibrium compared with the TOV equation.

| Quantity                   | Newtonian theory        | Relativistic TOV theory                          |
|----------------------------|-------------------------|--------------------------------------------------|
| Mass equation              | $dm/dr = 4\pi r^2 \rho$ | $dm/dr = 4\pi r^2 \rho$                          |
| Force density              | $-Gm\rho/r^2$           | $-(Gm\rho/r^2)\mathcal{I}\mathcal{S}\mathcal{G}$ |
| Inertial correction        | 1                       | $\mathcal{I} = 1 + P/(\rho c^2)$                 |
| Pressure as gravity source | absent                  | $\mathcal{S} = 1 + 4\pi r^3 P/(mc^2)$            |
| Curvature correction       | 1                       | $\mathcal{G} = (1 - 2Gm/(rc^2))^{-1}$            |
| Surface redshift           | $z \simeq GM/(Rc^2)$    | $z = (1 - 2GM/(Rc^2))^{-1/2} - 1$                |

**Figure 3.** Pressure profiles for the uniform-density Schwarzschild interior solution, normalized to the central pressure. As compactness increases, pressure support becomes increasingly centrally concentrated.

The TOV equation becomes

$$\frac{dP}{dr} = -\frac{G(\rho_0 + P/c^2)(4\pi\rho_0 r^3/3 + 4\pi r^3 P/c^2)}{r^2(1 - 8\pi G\rho_0 r^2/(3c^2))}. \quad (83)$$

It is convenient to introduce

$$q(r) = \frac{P(r)}{\rho_0 c^2}. \quad (84)$$

Using  $M = 4\pi\rho_0 R^3/3$ , the differential equation becomes

$$\frac{dq}{dr} = -\frac{4\pi G\rho_0 r(1+q)(1/3+q)}{c^2(1 - 2\beta r^2/R^2)}. \quad (85)$$

The exact solution satisfying  $P(R) = 0$  is

$$\frac{P(r)}{\rho_0 c^2} = \frac{\sqrt{1 - 2\beta r^2/R^2} - \sqrt{1 - 2\beta}}{3\sqrt{1 - 2\beta} - \sqrt{1 - 2\beta r^2/R^2}}. \quad (86)$$

The central pressure is therefore

$$\frac{P_c}{\rho_0 c^2} = \frac{1 - \sqrt{1 - 2\beta}}{3\sqrt{1 - 2\beta} - 1}. \quad (87)$$

The corresponding redshift factor inside the star is

$$e^{\Phi(r)} = \frac{1}{2} \left[ 3\sqrt{1 - 2\beta} - \sqrt{1 - 2\beta r^2/R^2} \right]. \quad (88)$$

**Proposition 1.** For an isotropic uniform-density relativistic star, the central pressure is finite only if

$$3\sqrt{1 - 2\beta} > 1. \quad (89)$$

Solving (89),

$$\sqrt{1 - 2\beta} > \frac{1}{3}, \quad (90)$$

so

$$1 - 2\beta > \frac{1}{9}, \quad (91)$$

and hence

$$\beta < \frac{4}{9}. \quad (92)$$

Thus

$$R > \frac{9GM}{4c^2}. \quad (93)$$

This is the Buchdahl bound for the uniform-density model and, under more general assumptions of isotropic pressure and nonincreasing density, for relativistic fluid spheres [5].

*Remark 5.* The Buchdahl radius  $R = 9GM/(4c^2)$  is larger than the Schwarzschild radius  $R_s = 2GM/c^2$ . A static perfect-fluid star cannot be compressed arbitrarily close to its event horizon without requiring divergent pressure.

The gravitational redshift of a photon emitted at the surface and observed at infinity is

$$1 + z_s = \left(1 - \frac{2GM}{Rc^2}\right)^{-1/2}, \quad (94)$$

or

$$z_s = (1 - 2\beta)^{-1/2} - 1. \quad (95)$$

At the Buchdahl limit, (95) gives

$$z_s < 2. \quad (96)$$

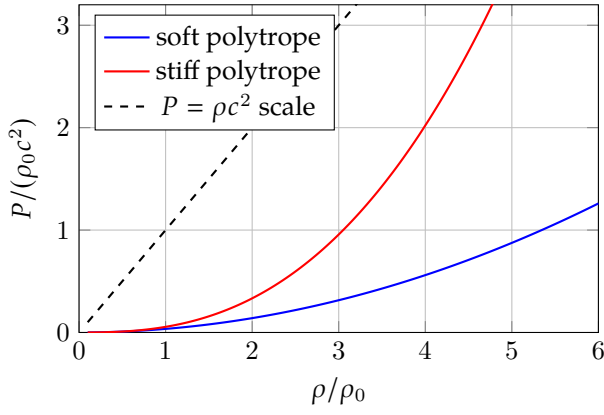
## 9. Equations of State and Maximum Mass

The TOV equation is not closed until an equation of state is supplied. For cold neutron-star matter one often writes

$$P = P(\epsilon), \quad \epsilon = \rho c^2. \quad (97)$$

A useful local representation is a polytrope,

$$P = K\rho_0^\Gamma, \quad (98)$$



**Figure 4.** Qualitative polytropic equations of state. Stiffer equations of state produce higher pressure at a given density and generally support larger maximum masses, but the slope must remain causal.

where  $\rho_0$  is rest-mass density,  $K$  is a stiffness parameter, and  $\Gamma$  is an adiabatic index. For a relativistic cold polytrope the total energy density may be approximated by

$$\epsilon = \rho_0 c^2 + \frac{P}{\Gamma - 1}. \quad (99)$$

The sound speed is

$$c_s^2 = \frac{dP}{d\epsilon}. \quad (100)$$

Causality requires

$$c_s^2 \leq c^2. \quad (101)$$

Equivalently,

$$\frac{dP}{d\rho} \leq c^2. \quad (102)$$

For the polytropic form (98) with (99), the derivative with respect to rest-mass density is

$$\frac{dP}{d\rho_0} = \Gamma K \rho_0^{\Gamma-1} = \Gamma \frac{P}{\rho_0}. \quad (103)$$

The corresponding energy-density derivative is

$$\frac{d\epsilon}{d\rho_0} = c^2 + \frac{\Gamma}{\Gamma - 1} \frac{P}{\rho_0}. \quad (104)$$

Thus

$$\frac{c_s^2}{c^2} = \frac{\Gamma P / (\rho_0 c^2)}{1 + \Gamma P / [(\Gamma - 1) \rho_0 c^2]}. \quad (105)$$

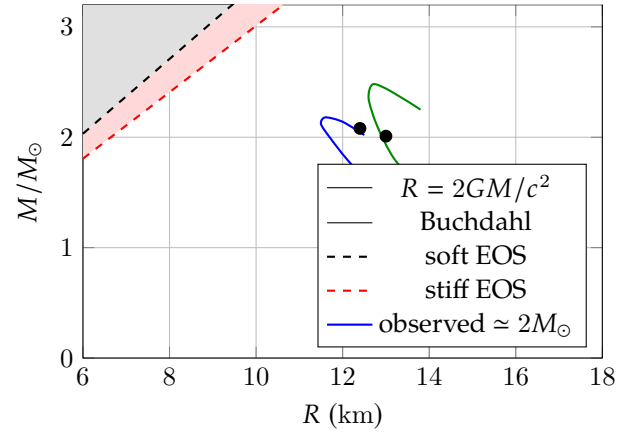
At high pressure this tends to

$$\lim_{P/(\rho_0 c^2) \rightarrow \infty} \frac{c_s^2}{c^2} = \Gamma - 1. \quad (106)$$

Therefore a single polytrope with  $\Gamma > 2$  cannot remain causal at arbitrarily high density, although it may still be a useful local fit over a finite interval.

The specific relativistic enthalpy is often useful:

$$h(P) = \int_0^P \frac{dP'}{\epsilon(P') + P'}. \quad (107)$$



**Figure 5.** Qualitative mass-radius curves for soft and stiff neutron-star equations of state. Each curve has a maximum mass. The shaded regions indicate the black-hole and Buchdahl exclusions. Observed two-solar-mass pulsars require the physical curve to reach at least about  $2 M_\odot$ .

Since  $\epsilon + P = \rho c^2 + P$ , the conservation equation implies

$$h(r) + \Phi(r) = \text{constant}. \quad (108)$$

This relation is frequently used in relativistic stellar modeling.

For a one-parameter sequence of nonrotating stars labeled by central density  $\rho_c$ , the gravitational mass is

$$M = M(\rho_c). \quad (109)$$

The onset of radial instability occurs at the first maximum:

$$\frac{dM}{d\rho_c} = 0. \quad (110)$$

Configurations with

$$\frac{dM}{d\rho_c} > 0 \quad (111)$$

are normally stable to the fundamental radial mode, while those beyond the maximum are unstable.

Modern observations require the maximum mass of the cold nonrotating sequence to exceed the masses of accurately measured pulsars. PSR J0348+0432 has

$$M = 2.01 \pm 0.04 M_\odot, \quad (112)$$

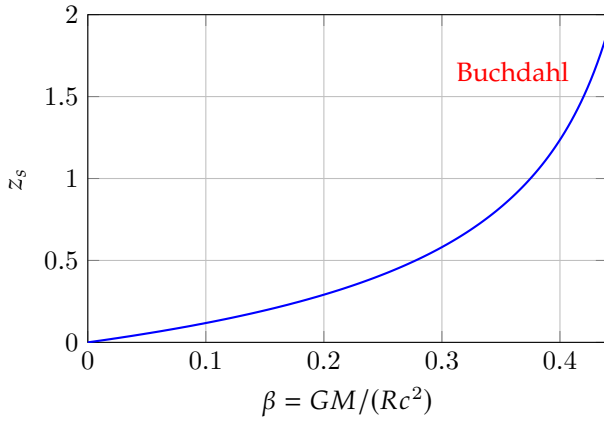
and PSR J0740+6620 has a mass near

$$M \simeq 2.08 M_\odot. \quad (113)$$

These measurements rule out equations of state that are too soft at supranuclear density [13, 14, 15]. Broadly, viable nonrotating maximum masses are expected to be of order

$$M_{\text{max}} \sim 2\text{--}3 M_\odot, \quad (114)$$

depending on the high-density equation of state [11, 12].



**Figure 6.** Surface gravitational redshift as a function of compactness. Relativistic redshift is small for ordinary stars but is a directly observable strong-field effect for neutron stars.

## 10. Numerical Integration of the TOV Equations

A numerical stellar model is obtained by specifying a central density or central pressure. The coupled first-order system is

$$\frac{dm}{dr} = 4\pi r^2 \rho(P), \quad (115)$$

$$\frac{dP}{dr} = -\frac{Gm\rho(P)}{r^2} \left(1 + \frac{P}{\rho(P)c^2}\right) \left(1 + \frac{4\pi r^3 P}{mc^2}\right) \left(1 - \frac{2Gm}{rc^2}\right)^{-1}. \quad (116)$$

The central boundary conditions are

$$m(0) = 0, \quad P(0) = P_c. \quad (117)$$

Because (116) contains  $m/r^2$ , integration is started at a small radius using series expansions. Near the center,

$$m(r) = \frac{4\pi}{3} \rho_c r^3 + O(r^5), \quad (118)$$

and

$$P(r) = P_c - \frac{2\pi G}{3} (\rho_c + P_c/c^2)(\rho_c + 3P_c/c^2)r^2 + O(r^4). \quad (119)$$

The surface is reached when

$$P(R) = 0. \quad (120)$$

Then

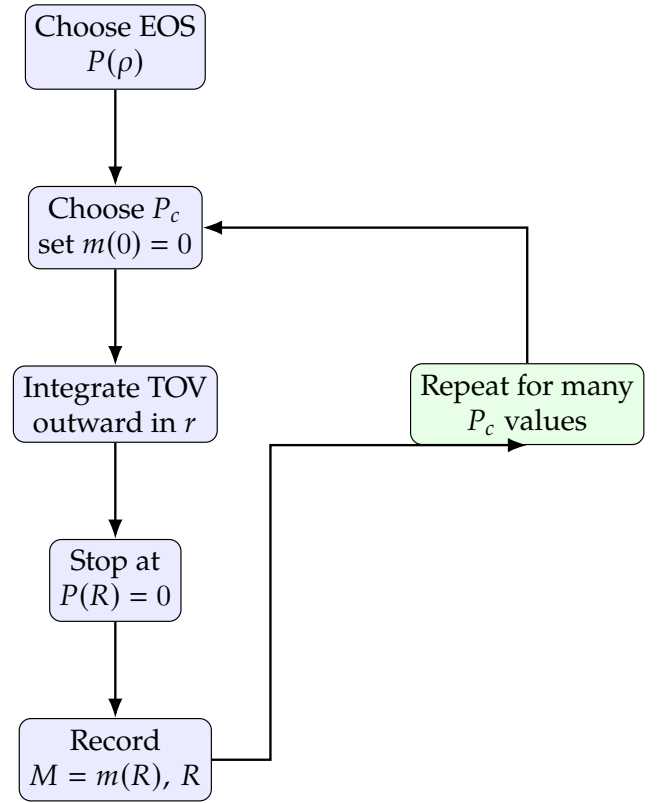
$$M = m(R). \quad (121)$$

The redshift potential may be integrated from

$$\frac{d\Phi}{dr} = -\frac{1}{\rho c^2 + P} \frac{dP}{dr}, \quad (122)$$

with the boundary condition

$$\Phi(R) = \frac{1}{2} \ln \left(1 - \frac{2GM}{Rc^2}\right). \quad (123)$$



**Figure 7.** Numerical construction of a mass-radius sequence. The central pressure labels one stellar model; varying it produces the equilibrium curve and reveals the maximum mass.

The baryonic mass is sometimes computed for comparison with the gravitational mass:

$$M_b = 4\pi \int_0^R \rho_0(r) \left(1 - \frac{2Gm(r)}{rc^2}\right)^{-1/2} r^2 dr. \quad (124)$$

The binding energy is then

$$E_b = (M_b - M)c^2. \quad (125)$$

For neutron stars,  $E_b$  is typically a substantial fraction of  $Mc^2$ , reflecting the strong gravity of the configuration.

## 11. Worked Numerical Applications

Useful constants are

$$\frac{GM_\odot}{c^2} = 1.4766 \text{ km}, \quad M_\odot = 1.9885 \times 10^{30} \text{ kg}. \quad (126)$$

### 11.1. Compactness and redshift of a massive neutron star

Consider a neutron star with

$$M = 2.08 M_\odot, \quad R = 12.4 \text{ km}. \quad (127)$$

Its compactness is

$$\beta = \frac{GM}{Rc^2} = \frac{1.4766 \text{ km}}{12.4 \text{ km}} (2.08) = 0.248. \quad (128)$$

The Schwarzschild radius is

$$R_s = 2\beta R = 6.15 \text{ km.} \quad (129)$$

The surface redshift is

$$\begin{aligned} z_s &= (1 - 2\beta)^{-1/2} - 1 \\ &= (1 - 0.496)^{-1/2} - 1 = 0.409. \end{aligned} \quad (130)$$

Thus a photon emitted with energy  $E_e$  at the surface is observed at infinity with

$$E_\infty = \frac{E_e}{1 + z_s} = 0.710 E_e. \quad (131)$$

The mean density is

$$\begin{aligned} \bar{\rho} &= \frac{3M}{4\pi R^3} \\ &= \frac{3(2.08)(1.9885 \times 10^{30} \text{ kg})}{4\pi(1.24 \times 10^4 \text{ m})^3} \\ &= 5.18 \times 10^{17} \text{ kg m}^{-3}. \end{aligned} \quad (132)$$

The relativistic surface gravity measured by a static local observer is

$$g_s = \frac{GM}{R^2} \left( 1 - \frac{2GM}{Rc^2} \right)^{-1/2}. \quad (133)$$

For the same star,

$$g_s = 2.53 \times 10^{12} \text{ m s}^{-2}. \quad (134)$$

## 11.2. Central pressure in the uniform-density model

Take

$$M = 1.40 M_\odot, \quad R = 12.0 \text{ km.} \quad (135)$$

Then

$$\beta = \frac{1.4766(1.40)}{12.0} = 0.172. \quad (136)$$

The mean density, equal to the constant density in this model, is

$$\rho_0 = 3.85 \times 10^{17} \text{ kg m}^{-3}. \quad (137)$$

Since

$$\sqrt{1 - 2\beta} = 0.810, \quad (138)$$

equation (87) gives

$$\frac{P_c}{\rho_0 c^2} = \frac{1 - 0.810}{3(0.810) - 1} = 0.133. \quad (139)$$

Thus

$$P_c = 0.133 \rho_0 c^2 = 4.61 \times 10^{33} \text{ Pa.} \quad (140)$$

This pressure is enormous by terrestrial standards but typical of neutron-star interiors.

## 11.3. Magnitude of TOV correction factors

At a radius where

$$\frac{P}{\rho c^2} = 0.10, \quad \frac{4\pi r^3 P}{mc^2} = 0.20, \quad \frac{2Gm}{rc^2} = 0.35, \quad (141)$$

the correction factors are

$$I = 1.10, \quad S = 1.20, \quad \mathcal{G} = 1.54. \quad (142)$$

Their product is

$$I S \mathcal{G} = 2.03. \quad (143)$$

The required pressure gradient is therefore about twice the Newtonian value at the same  $m$ ,  $r$ , and  $\rho$ . Relativistic stars are harder to support than Newtonian intuition suggests.

## 11.4. Binding energy of a uniform-density star

The uniform-density solution also gives a transparent estimate of the binding energy. If the rest-mass density is identified with the constant density  $\rho_0$ , the baryonic mass is

$$M_b = 4\pi\rho_0 \int_0^R \frac{r^2 dr}{\sqrt{1 - 2\beta r^2/R^2}}. \quad (144)$$

With  $y = r/R$ , this becomes

$$\frac{M_b}{M} = 3 \int_0^1 \frac{y^2 dy}{\sqrt{1 - 2\beta y^2}}. \quad (145)$$

The integral has the closed form

$$\frac{M_b}{M} = \frac{3}{2(2\beta)^{3/2}} \left[ \sin^{-1} \sqrt{2\beta} - \sqrt{2\beta(1 - 2\beta)} \right]. \quad (146)$$

For the  $1.40 M_\odot$ ,  $12.0 \text{ km}$  example,  $\beta = 0.172$ , and (146) gives

$$\frac{M_b}{M} = 1.135. \quad (147)$$

Thus

$$M_b - M = 0.189 M_\odot. \quad (148)$$

The corresponding binding energy is

$$E_b = 0.189 M_\odot c^2 = 3.38 \times 10^{46} \text{ J.} \quad (149)$$

This is about

$$\frac{E_b}{Mc^2} = 0.135 \quad (150)$$

of the gravitational mass energy. The estimate is model dependent because realistic stars are centrally condensed and have internal energy contributions to  $\rho$ , but it correctly shows that neutron-star binding energies are astrophysically enormous.

## 12. Extensions and Observational Tests

The TOV equation is the nonrotating baseline for a broader hierarchy of relativistic stellar models. Several extensions are important in current compact-object astrophysics.

First, rotation modifies both the equilibrium shape and the exterior multipole moments. In the slow-rotation limit, the stellar angular momentum is written

$$J = I\Omega, \quad (151)$$

where  $I$  is the moment of inertia and  $\Omega$  is the angular velocity measured at infinity. Frame dragging introduces an angular velocity  $\omega(r)$  of local inertial frames. To first order in  $\Omega$ , the fluid angular velocity relative to local inertial frames is

$$\bar{\omega}(r) = \Omega - \omega(r). \quad (152)$$

The moment of inertia is approximately constrained by mass and radius, and a measurement of  $I$  in a binary pulsar would strongly restrict the equation of state.

Second, magnetic fields and elastic stresses can make the pressure anisotropic. A simple anisotropic stress tensor has radial and tangential pressures  $P_r$  and  $P_t$ . The hydrostatic equation generalizes schematically to

$$\frac{dP_r}{dr} = -\frac{Gm\rho}{r^2} \left(1 + \frac{P_r}{\rho c^2}\right) \left(1 + \frac{4\pi r^3 P_r}{mc^2}\right) \left(1 - \frac{2Gm}{rc^2}\right)^{-1} + \frac{2(P_t - P_r)}{r}. \quad (153)$$

The final term is not a relativistic correction to perfect-fluid equilibrium; it is a force caused by anisotropic stress. Positive  $P_t - P_r$  supplies additional outward support, while negative  $P_t - P_r$  has the opposite effect.

Third, binary inspirals probe the response of a star to an external tidal field. The leading dimensionless tidal deformability is

$$\Lambda_{\text{tidal}} = \frac{2}{3} k_2 \left( \frac{Rc^2}{GM} \right)^5, \quad (154)$$

where  $k_2$  is the quadrupolar Love number. The strong fifth-power dependence on  $R/M$  means that gravitational waves are especially sensitive to radii. The binary neutron-star event GW170817 constrained tidal deformabilities and therefore disfavored extremely large radii for canonical-mass neutron stars [16]. In practice, a candidate equation of state is tested by solving the TOV equations, computing  $M(R)$  and tidal response, and comparing the predicted observables with pulsar timing, X-ray pulse profiles, and gravitational-wave data.

Finally, surface redshift remains a direct compactness observable when identifiable spectral features are present. A measured redshift determines

$$\frac{GM}{Rc^2} = \frac{1}{2} \left[ 1 - \frac{1}{(1 + z_s)^2} \right]. \quad (155)$$

It does not separately determine  $M$  and  $R$ , but combined with an independent mass or radius measurement it becomes a stringent test of relativistic stellar structure.

## 13. Discussion

The derivation has assumed a static, spherical, isotropic perfect fluid. These assumptions are restrictive but powerful. They exclude rotation, magnetic fields, elastic crustal shear, superfluid multifluid dynamics, temperature gradients, and time dependence. Slowly rotating neutron stars can be treated perturbatively through the Hartle–Thorne formalism, while rapidly rotating stars require two-dimensional numerical relativity. Strong magnetic fields can introduce anisotropic stresses, and thermal effects are essential in proto-neutron stars and merger remnants. Nevertheless, the TOV equation remains the baseline against which these refinements are measured.

The most important conceptual difference between Newtonian and relativistic equilibrium is the role of pressure. In Newtonian theory pressure appears only through its gradient. A spatially uniform pressure has no gravitational effect. In general relativity pressure contributes to  $T^\mu_\nu$ , and hence to curvature. This contribution appears twice in the TOV equation: pressure increases the inertial density through  $\rho + P/c^2$ , and it increases the active gravitational mass through  $m + 4\pi r^3 P/c^2$ . A third correction arises from spatial curvature, represented by  $(1 - 2Gm/(rc^2))^{-1}$ . These effects all increase the magnitude of the pressure gradient needed for equilibrium.

The Buchdahl bound illustrates the consequence. Even for the artificially favorable case of uniform density, the central pressure diverges at  $R = 9GM/(4c^2)$ . Realistic equations of state do not have exactly uniform density, and causality imposes additional restrictions. The existence of a maximum mass follows physically because increasing central density strengthens gravity faster than pressure support can compensate once the configuration becomes sufficiently compact. The maximum mass is not a universal constant; it depends on the high-density nuclear equation of state. Observed pulsars near  $2M_\odot$  imply that cold dense matter must be sufficiently stiff above nuclear saturation density, but causality and tidal-deformability constraints limit how stiff it can be [12, 15].

Mass-radius observations, pulse-profile modeling, gravitational waves from binary neutron-star mergers, and nuclear experiments are now jointly constraining  $P(\rho)$  at supranuclear density. The TOV equation is the bridge between microphysics and observables: for a proposed equation of state it predicts  $M(R)$ , compactness, redshift, tidal deformability, and stability. Conversely, observations restrict the allowed pressure at densities inaccessible to terrestrial laboratories.

## 14. Conclusion

Starting from the static spherical metric, the perfect-fluid stress-energy tensor, Einstein's equations, and local conservation of stress energy, we have derived the

TOV equation in its standard factored form:

$$\frac{dP}{dr} = -\frac{Gm\rho}{r^2} \left(1 + \frac{P}{\rho c^2}\right) \left(1 + \frac{4\pi r^3 P}{mc^2}\right) \left(1 - \frac{2Gm}{rc^2}\right)^{-1}. \quad (156)$$

The derivation shows that relativistic hydrostatic equilibrium is not Newtonian equilibrium with small corrections appended, but a geometrical condition tying pressure gradients to spacetime curvature. The first correction factor represents pressure inertia, the second pressure as a source of gravity, and the third spatial curvature. In the weak-field, low-pressure limit the equation reduces to Newtonian hydrostatic balance.

The uniform-density Schwarzschild interior solution demonstrates explicitly how relativistic pressure grows with compactness and leads to the Buchdahl bound,

$$R > \frac{9GM}{4c^2}. \quad (157)$$

For neutron stars, the TOV equation, combined with a causal high-density equation of state, predicts mass-radius sequences with a maximum mass. Observed pulsars with masses near  $2M_\odot$  require that the equation of state be sufficiently stiff, while causality and compactness restrict the range of possible radii and maximum masses. Thus the TOV equation remains one of the essential tools connecting general relativity, nuclear physics, and astronomical observation.

## A. Algebraic Details for the Uniform-Density Solution

This appendix gives the intermediate integration steps leading to (86). Let

$$x = \frac{r}{R}, \quad A(x) = \sqrt{1 - 2\beta x^2}, \quad A_s = \sqrt{1 - 2\beta}. \quad (158)$$

For uniform density, (85) may be written

$$\frac{dq}{(1+q)(1/3+q)} = -\frac{3\beta x dx}{1 - 2\beta x^2}. \quad (159)$$

The left side decomposes as

$$\frac{1}{(1+q)(1/3+q)} = -\frac{3}{2} \frac{1}{1+q} + \frac{3}{2} \frac{1}{1/3+q}. \quad (160)$$

Therefore

$$\int \frac{dq}{(1+q)(1/3+q)} = \frac{3}{2} \ln \left( \frac{q+1/3}{q+1} \right). \quad (161)$$

The right side integrates to

$$-3\beta \int \frac{x dx}{1 - 2\beta x^2} = \frac{3}{4} \ln(1 - 2\beta x^2) + C. \quad (162)$$

Combining these results gives

$$\ln \left( \frac{q+1/3}{q+1} \right) = \frac{1}{2} \ln(1 - 2\beta x^2) + C_1. \quad (163)$$

Exponentiating,

$$\frac{q+1/3}{q+1} = C_2 A(x). \quad (164)$$

The surface condition  $q(1) = 0$  fixes

$$C_2 = \frac{1}{3A_s}. \quad (165)$$

Thus

$$3A_s(q+1/3) = A(x)(q+1). \quad (166)$$

Solving this linear equation for  $q$  gives

$$q(x) = \frac{A(x) - A_s}{3A_s - A(x)}, \quad (167)$$

which is (86).

## B. Central Series Expansion

The series expansion used to start numerical integrations follows directly from the TOV equation. Suppose

$$\rho(r) = \rho_c + \rho_2 r^2 + O(r^4), \quad P(r) = P_c + P_2 r^2 + O(r^4). \quad (168)$$

Then the mass equation gives

$$m(r) = \frac{4\pi}{3} \rho_c r^3 + O(r^5). \quad (169)$$

Substitution into the unfactored TOV equation gives, to leading order,

$$\frac{dP}{dr} = -\frac{4\pi G}{3} (\rho_c + P_c/c^2)(\rho_c + 3P_c/c^2)r + O(r^3). \quad (170)$$

Since  $dP/dr = 2P_2 r + O(r^3)$ , one finds

$$P_2 = -\frac{2\pi G}{3} (\rho_c + P_c/c^2)(\rho_c + 3P_c/c^2). \quad (171)$$

This reproduces (119) and shows explicitly why  $P'(0) = 0$  while  $P''(0) < 0$  for positive density and pressure.

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