



RELATIVITY AND BLACK HOLES

Derivation of the Schwarzschild Metric and the Analysis of Geodesic Motion near the Event Horizon

Academic Research Paper

Theoretical Astrophysics — Step-by-Step Analytic Derivations, Vector Diagrams and Numerical Tables

Derivation of the Schwarzschild Metric and the Analysis of Geodesic Motion near the Event Horizon

Relativity and Black Holes

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Abstract. The Schwarzschild solution is the unique asymptotically flat, spherically symmetric vacuum geometry of general relativity and is the mathematical foundation of the modern theory of black holes, relativistic stellar structure, compact-object dynamics, and strong-field lensing. This paper derives the Schwarzschild metric from the static spherically symmetric line element with two unknown radial functions, computes the nonzero Christoffel symbols and Ricci tensor components explicitly, and solves the vacuum Einstein equations step by step. The integration constant is fixed by the Newtonian weak-field limit, yielding the Schwarzschild radius $r_s = 2GM/c^2$. We then analyze geodesic motion by deriving conserved energy and angular momentum from the geodesic Lagrangian, constructing effective potentials for massive particles and photons, and identifying circular orbits, the innermost stable circular orbit, the marginally bound orbit, the photon sphere, and the black-hole shadow scale. The coordinate singularity at the event horizon is interpreted through radial infall in proper and coordinate time and removed using ingoing Eddington–Finkelstein coordinates. Observable tests, including gravitational redshift, perihelion precession, light deflection, Shapiro delay, compact-star redshift estimates, and Event Horizon Telescope angular scales, are discussed quantitatively.

Keywords: Schwarzschild metric; black holes; geodesics; event horizon; photon sphere; gravitational redshift; relativistic orbits

1. Introduction

The Schwarzschild metric occupies a central position in relativistic astrophysics. It is the first exact solution of Einstein’s field equations, found in 1916 within months of the final formulation of general relativity [1, 2]. Despite its simple appearance, it encodes many of the characteristic phenomena of strong gravity: gravitational time dilation, event horizons, relativistic orbital precession, unstable photon orbits, and the optical appearance of black holes. It is also the exterior spacetime of any isolated, nonrotating, spherically symmetric body, irrespective of the body’s internal composition or dynamical radial pulsation, as formalized by Birkhoff’s theorem [3, 5, 8].

In Newtonian gravity, the exterior gravitational field of a spherical body is completely determined by its mass. General relativity preserves this result in a deeper geometrical form: outside the matter distribution, spherical symmetry and the vacuum Einstein equations determine the spacetime metric uniquely up to one mass parameter. The Newtonian potential $\Phi = -GM/r$ is recovered only in the weak-field, slow-motion limit. The exact relativistic result replaces the Newtonian scalar potential by a curved Lorentzian geometry with line element

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) c^2 dt^2 + \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \quad (1)$$

where

$$r_s = \frac{2GM}{c^2}, \quad d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2. \quad (2)$$

The coordinate radius $r = r_s$ is the Schwarzschild event horizon. For a solar-mass object, $r_s \simeq 2.95$ km; for the

black hole in M87, with mass $M \simeq 6.5 \times 10^9 M_\odot$, the corresponding length scale is astronomical.

The purpose of this paper is to give a complete derivation and physical analysis appropriate for advanced undergraduate and graduate study. We proceed from the most general static spherically symmetric ansatz in curvature coordinates, compute the connection and curvature explicitly, solve the vacuum field equations, and then use the resulting geometry to study geodesic motion. Particular emphasis is placed on maintaining units, distinguishing coordinate effects from invariant physics, and connecting mathematical results to observations.

The Schwarzschild solution is not only a historical milestone. It provides the local model for compact-object environments, accretion flows, relativistic jets, stellar dynamics near massive black holes, gravitational lensing, and tests of relativistic gravity [7, 6, 9, 10]. Although astrophysical black holes rotate and are therefore described more accurately by the Kerr metric, the Schwarzschild geometry remains the essential first approximation and the conceptual basis for understanding event horizons and photon capture.

2. Physical and Mathematical Framework

2.1. Einstein equations in vacuum

General relativity relates geometry to matter through Einstein’s field equations,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}. \quad (3)$$

In the exterior of an isolated spherical body, the stress-energy tensor is zero:

$$T_{\mu\nu} = 0. \quad (4)$$

Taking the trace of (3) in four dimensions gives

$$G^\mu{}_\mu = R - 2R = -R = \frac{8\pi G}{c^4} T^\mu{}_\mu. \quad (5)$$

Thus, in vacuum,

$$R = 0, \quad (6)$$

and the field equations reduce to

$$R_{\mu\nu} = 0. \quad (7)$$

The task is therefore to find all static spherically symmetric metrics whose Ricci tensor vanishes.

Definition 1. A spacetime is *static* if it possesses a time-like Killing vector field that is hypersurface orthogonal. It is *spherically symmetric* if the rotation group $SO(3)$ acts by isometries with two-dimensional spacelike orbits.

Staticity excludes rotation and time-dependent gravitational radiation. Spherical symmetry implies that the angular part of the metric is proportional to the metric on the unit two-sphere. We choose the radial coordinate r so that spheres of constant t and r have area $4\pi r^2$. This is the areal radius, not necessarily the proper radial distance.

2.2. Static spherically symmetric ansatz

The most general static spherically symmetric metric in areal coordinates can be written as

$$ds^2 = -A(r)c^2 dt^2 + B(r)dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2. \quad (8)$$

The functions $A(r)$ and $B(r)$ are assumed positive in the exterior static region. Asymptotic flatness requires

$$\lim_{r \rightarrow \infty} A(r) = 1, \quad \lim_{r \rightarrow \infty} B(r) = 1. \quad (9)$$

The metric components are

$$g_{\mu\nu} = \text{diag}(-Ac^2, B, r^2, r^2 \sin^2 \theta), \quad (10)$$

and the inverse metric is

$$g^{\mu\nu} = \text{diag}\left(-\frac{1}{Ac^2}, \frac{1}{B}, \frac{1}{r^2}, \frac{1}{r^2 \sin^2 \theta}\right). \quad (11)$$

Primes will denote differentiation with respect to r .

The Christoffel symbols are determined by

$$\Gamma^\alpha{}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} (\partial_\mu g_{\beta\nu} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu}). \quad (12)$$

Because the metric is diagonal and independent of t and ϕ , most connection coefficients vanish.

3. Connection and Curvature of the Ansatz

3.1. Christoffel symbols

For the temporal components, the only radial derivative is

$$\partial_r g_{tt} = -c^2 A'(r). \quad (13)$$

Thus

$$\Gamma^t{}_{tr} = \Gamma^t{}_{rt} = \frac{1}{2} g^{tt} \partial_r g_{tt} = \frac{A'}{2A}, \quad (14)$$

$$\Gamma^r{}_{tt} = -\frac{1}{2} g^{rr} \partial_r g_{tt} = \frac{c^2 A'}{2B}. \quad (15)$$

The purely radial coefficient is

$$\Gamma^r{}_{rr} = \frac{1}{2} g^{rr} \partial_r g_{rr} = \frac{B'}{2B}. \quad (16)$$

The angular radial couplings follow from $\partial_r g_{\theta\theta} = 2r$ and $\partial_r g_{\phi\phi} = 2r \sin^2 \theta$:

$$\Gamma^r{}_{\theta\theta} = -\frac{r}{B}, \quad (17)$$

$$\Gamma^r{}_{\phi\phi} = -\frac{r \sin^2 \theta}{B}, \quad (18)$$

$$\Gamma^\theta{}_{r\theta} = \Gamma^\theta{}_{\theta r} = \frac{1}{r}, \quad (19)$$

$$\Gamma^\phi{}_{r\phi} = \Gamma^\phi{}_{\phi r} = \frac{1}{r}. \quad (20)$$

Finally, the intrinsic connection on the unit sphere gives

$$\Gamma^\theta{}_{\phi\phi} = -\sin \theta \cos \theta, \quad (21)$$

$$\Gamma^\phi{}_{\theta\theta} = \Gamma^\phi{}_{\phi\theta} = \cot \theta. \quad (22)$$

Equations (14)–(22) are the nonzero independent Christoffel symbols.

Remark 1. The factors of c enter only in components involving t . They cancel out of most curvature equations because c is constant; its main role is dimensional, ensuring that t has units of time rather than length.

3.2. Ricci tensor components

The Ricci tensor is

$$R_{\mu\nu} = \partial_\alpha \Gamma^\alpha{}_{\mu\nu} - \partial_\nu \Gamma^\alpha{}_{\mu\alpha} + \Gamma^\alpha{}_{\mu\nu} \Gamma^\beta{}_{\alpha\beta} - \Gamma^\beta{}_{\mu\alpha} \Gamma^\alpha{}_{\nu\beta}. \quad (23)$$

We compute the independent components needed for the vacuum equations.

For R_{tt} , time derivatives vanish. The first term is

$$\partial_r \Gamma^r{}_{tt} = \frac{c^2}{2} \left(\frac{A''}{B} - \frac{A'B'}{B^2} \right). \quad (24)$$

The trace entering the third term is

$$\Gamma^\beta{}_{r\beta} = \frac{A'}{2A} + \frac{B'}{2B} + \frac{2}{r}. \quad (25)$$

Hence

$$\Gamma^r{}_{tt} \Gamma^\beta{}_{r\beta} = \frac{c^2 A'}{2B} \left(\frac{A'}{2A} + \frac{B'}{2B} + \frac{2}{r} \right). \quad (26)$$

The fourth term contains the two equal contributions involving Γ^t_{tr} and Γ^r_{tt} :

$$-\Gamma^\beta_{t\alpha}\Gamma^\alpha_{t\beta} = -\frac{c^2 A'^2}{2AB}. \quad (27)$$

Combining gives

$$R_{tt} = \frac{c^2}{2B} \left[A'' - \frac{A'B'}{2B} - \frac{A'^2}{2A} + \frac{2A'}{r} \right]. \quad (28)$$

For R_{rr} , one obtains

$$R_{rr} = -\frac{A''}{2A} + \frac{A'^2}{4A^2} + \frac{A'B'}{4AB} + \frac{B'}{Br}. \quad (29)$$

The angular component $R_{\theta\theta}$ is

$$R_{\theta\theta} = 1 - \frac{1}{B} + \frac{rB'}{2B^2} - \frac{rA'}{2AB}. \quad (30)$$

The remaining angular component is related by spherical symmetry:

$$R_{\phi\phi} = \sin^2 \theta R_{\theta\theta}. \quad (31)$$

Equations (28), (29), and (30) constitute the essential curvature content of the ansatz.

4. Solving the Vacuum Field Equations

4.1. Derivation of $A(r)B(r) = 1$

The vacuum conditions are

$$R_{tt} = 0, \quad R_{rr} = 0, \quad R_{\theta\theta} = 0. \quad (32)$$

A particularly efficient combination is

$$\frac{R_{tt}}{Ac^2} + \frac{R_{rr}}{B} = 0. \quad (33)$$

Substitution of (28) and (29) gives cancellation of the second-derivative and quadratic terms:

$$0 = \frac{1}{Br} \left(\frac{A'}{A} + \frac{B'}{B} \right) \quad (34)$$

$$= \frac{1}{Br} \frac{d}{dr} \ln(AB). \quad (35)$$

Since B and r are nonzero in the exterior region,

$$\frac{d}{dr} \ln(AB) = 0. \quad (36)$$

Therefore

$$A(r)B(r) = C_1, \quad (37)$$

where C_1 is constant. Asymptotic flatness (9) requires $C_1 = 1$, so

$$A(r)B(r) = 1, \quad B(r) = \frac{1}{A(r)}. \quad (38)$$

This result has a simple interpretation: in Schwarzschild coordinates, the gravitational redshift factor and the radial stretching factor are reciprocal in vacuum.

4.2. Derivation of $A(r) = 1 - r_s/r$

Using $B = 1/A$ in (30), we compute

$$B' = -\frac{A'}{A^2}. \quad (39)$$

Then

$$0 = R_{\theta\theta} = 1 - A + \frac{r}{2} \left(-\frac{A'}{A^2} \right) A^2 - \frac{rA'}{2} \quad (40)$$

$$= 1 - A - rA'. \quad (41)$$

Thus

$$rA' + A = 1. \quad (42)$$

This is equivalent to

$$\frac{d}{dr} (rA) = 1. \quad (43)$$

Integrating,

$$rA = r + C_2, \quad (44)$$

or

$$A(r) = 1 + \frac{C_2}{r}. \quad (45)$$

The constant C_2 is fixed by matching to the Newtonian limit.

In the weak-field, slow-motion limit, the time-time component of the metric is related to the Newtonian potential by

$$g_{tt} \simeq -\left(1 + \frac{2\Phi}{c^2}\right) c^2. \quad (46)$$

Since $g_{tt} = -Ac^2$, we require

$$A(r) \simeq 1 + \frac{2\Phi}{c^2}. \quad (47)$$

For a central mass M ,

$$\Phi(r) = -\frac{GM}{r}. \quad (48)$$

Hence

$$A(r) \simeq 1 - \frac{2GM}{c^2 r}. \quad (49)$$

Comparison with (45) gives

$$C_2 = -\frac{2GM}{c^2}. \quad (50)$$

Defining the Schwarzschild radius

$$r_s = \frac{2GM}{c^2}, \quad (51)$$

we obtain

$$A(r) = 1 - \frac{r_s}{r}, \quad B(r) = \left(1 - \frac{r_s}{r}\right)^{-1}. \quad (52)$$

Substitution into (8) gives the Schwarzschild line element,

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) c^2 dt^2 + \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 + r^2 d\Omega^2. \quad (53)$$

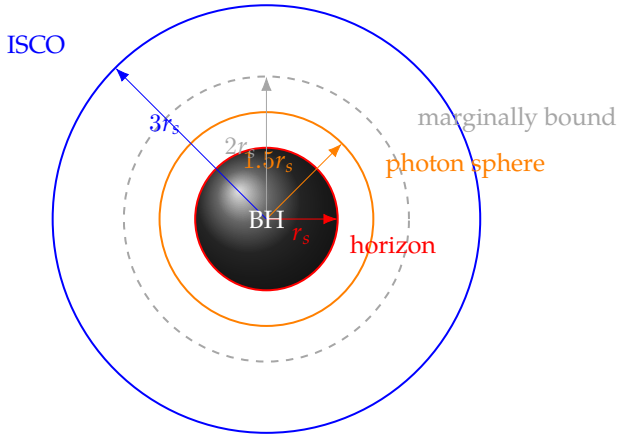


Figure 1. Characteristic radii of the Schwarzschild geometry in units of $r_s = 2GM/c^2$. The event horizon is at $r = r_s$, the photon sphere at $r = 1.5r_s$, the marginally bound timelike circular orbit at $r = 2r_s$, and the innermost stable circular orbit at $r = 3r_s$.

Theorem 1 (Birkhoff's theorem). *Every spherically symmetric solution of the vacuum Einstein equations is locally isometric to a portion of the Schwarzschild solution. Consequently, the exterior gravitational field of a spherically symmetric body is static even if the body undergoes radial pulsations.*

Remark 2. Birkhoff's theorem has no Newtonian analogue involving spacetime geometry, but it generalizes the shell theorem. It also implies that a perfectly spherical stellar collapse does not emit gravitational waves: gravitational radiation requires a time-dependent quadrupole or higher multipole moment.

5. Gravitational Time Dilation and Redshift

For a stationary observer at fixed r, θ, ϕ , the line element satisfies

$$ds^2 = -c^2 d\tau^2 = -\left(1 - \frac{r_s}{r}\right) c^2 dt^2. \quad (54)$$

Therefore the proper time interval measured by the local observer is

$$d\tau = \sqrt{1 - \frac{r_s}{r}} dt. \quad (55)$$

Clocks deeper in the gravitational field run more slowly relative to the coordinate time normalized at infinity.

Consider two static observers, one at radius r_e emitting light and another at radius r_o observing it. The conserved photon energy associated with the timelike Killing vector implies

$$h\nu(r) = \frac{E_\infty}{\sqrt{1 - r_s/r}}. \quad (56)$$

Thus

$$\frac{\nu_o}{\nu_e} = \sqrt{\frac{1 - r_s/r_e}{1 - r_s/r_o}}. \quad (57)$$

For observation at infinity, $r_o \rightarrow \infty$, this becomes

$$\frac{\nu_\infty}{\nu_e} = \sqrt{1 - \frac{r_s}{r_e}}. \quad (58)$$

The gravitational redshift is

$$1 + z = \frac{\nu_e}{\nu_\infty} = \left(1 - \frac{r_s}{r_e}\right)^{-1/2}. \quad (59)$$

As $r_e \rightarrow r_s^+$, $z \rightarrow \infty$. This divergence refers to photons emitted by static observers, whose proper acceleration also diverges at the horizon. It does not mean that a freely falling observer encounters a local singularity there.

The four-velocity of a static observer is fixed by $u^\mu u_\mu = -c^2$ and has only a time component:

$$u^t = \frac{1}{\sqrt{1 - r_s/r}}, \quad u^r = u^\theta = u^\phi = 0. \quad (60)$$

The observer is not in geodesic motion. Its four-acceleration is

$$a^\mu = u^\nu \nabla_\nu u^\mu. \quad (61)$$

Using (15), with $A = 1 - r_s/r$ and $B = 1/A$, the only nonzero component is

$$a^r = \Gamma^r_{tt} (u^t)^2 = \frac{c^2 r_s}{2r^2}. \quad (62)$$

The invariant magnitude measured by an accelerometer carried by the observer is

$$a = \sqrt{g_{rr} a^r a^r} = \frac{GM}{r^2 \sqrt{1 - r_s/r}}. \quad (63)$$

At large radius this becomes the Newtonian value GM/r^2 , while it diverges at the horizon. Thus hovering just above a Schwarzschild horizon requires unbounded rocket acceleration even though freely falling frames remain locally regular.

6. Geodesic Motion

6.1. Lagrangian and constants of motion

Geodesics extremize the proper time for timelike curves or an affine parameter for null curves. In the equatorial plane $\theta = \pi/2$, allowed by spherical symmetry, the Schwarzschild line element gives the Lagrangian

$$2\mathcal{L} = -\left(1 - \frac{r_s}{r}\right) c^2 \dot{t}^2 + \left(1 - \frac{r_s}{r}\right)^{-1} \dot{r}^2 + r^2 \dot{\phi}^2, \quad (64)$$

where a dot denotes differentiation with respect to proper time τ for massive particles, or an affine parameter λ for photons. The normalization condition is

$$2\mathcal{L} = -\kappa c^2, \quad \kappa = \begin{cases} 1, & \text{massive particle,} \\ 0, & \text{photon.} \end{cases} \quad (65)$$

Because t and ϕ are cyclic coordinates, there are two conserved quantities. For massive particles, the conserved energy per unit rest mass is

$$E = \left(1 - \frac{r_s}{r}\right) c^2 \dot{t}, \quad (66)$$

and the conserved angular momentum per unit rest mass is

$$L = r^2 \dot{\phi}. \quad (67)$$

Substitution into (65) yields

$$-\frac{E^2}{c^2(1 - r_s/r)} + \left(1 - \frac{r_s}{r}\right)^{-1} \dot{r}^2 + \frac{L^2}{r^2} = -c^2. \quad (68)$$

Multiplying by $1 - r_s/r$ gives the radial equation

$$\dot{r}^2 + V_{\text{eff}}(r) = \frac{E^2}{c^2}, \quad (69)$$

with effective potential

$$V_{\text{eff}}(r) = \left(1 - \frac{r_s}{r}\right) \left(c^2 + \frac{L^2}{r^2}\right). \quad (70)$$

The term $c^2(1 - r_s/r)$ is the relativistic gravitational contribution, while the angular momentum term is modified by the same redshift factor. This modification produces the inner unstable region absent in Newtonian gravity.

For photons, define conserved quantities

$$E_\gamma = \left(1 - \frac{r_s}{r}\right) c^2 \frac{dt}{d\lambda}, \quad (71)$$

$$L_\gamma = r^2 \frac{d\phi}{d\lambda}. \quad (72)$$

The null condition gives

$$\left(\frac{dr}{d\lambda}\right)^2 + V_{\text{ph}}(r) = \frac{E_\gamma^2}{c^2}, \quad (73)$$

where

$$V_{\text{ph}}(r) = \left(1 - \frac{r_s}{r}\right) \frac{L_\gamma^2}{r^2}. \quad (74)$$

6.2. Circular massive orbits and the ISCO

A circular timelike orbit satisfies

$$\dot{r} = 0, \quad \frac{dV_{\text{eff}}}{dr} = 0. \quad (75)$$

Differentiate (70):

$$\frac{dV_{\text{eff}}}{dr} = \frac{r_s}{r^2} \left(c^2 + \frac{L^2}{r^2}\right) - \left(1 - \frac{r_s}{r}\right) \frac{2L^2}{r^3}. \quad (76)$$

Setting this to zero and multiplying by r^3 gives

$$r_s c^2 r + \frac{r_s L^2}{r} - 2L^2 + \frac{2r_s L^2}{r} = 0. \quad (77)$$

Thus

$$L^2 = \frac{c^2 r_s r^2}{2r - 3r_s}. \quad (78)$$

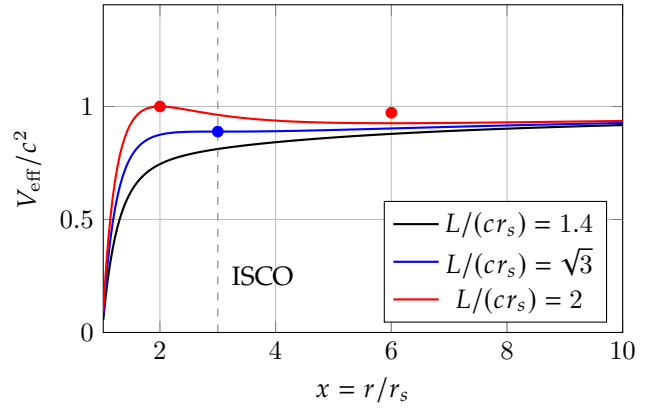


Figure 2. Massive-particle effective potential (70) in dimensionless form. For $L/(cr_s) = 2$ there are an unstable circular orbit at $r = 2r_s$ and a stable one at $r = 6r_s$. The limiting case $L/(cr_s) = \sqrt{3}$ has an inflection point at the ISCO, $r = 3r_s$.

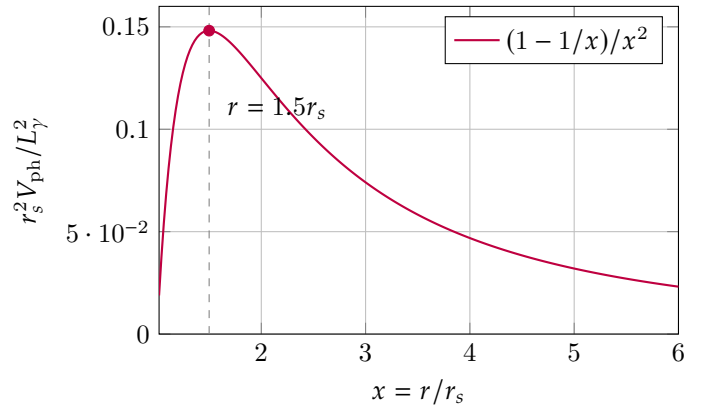


Figure 3. Photon effective potential (74). Its maximum occurs at the photon sphere $r = 3r_s/2$, corresponding to an unstable circular null orbit. Photons with smaller impact parameter are captured.

Circular timelike orbits require

$$r > \frac{3}{2} r_s. \quad (79)$$

Using (69), the corresponding energy is

$$\frac{E^2}{c^4} = \frac{(r - r_s)^2}{r(r - \frac{3}{2} r_s)}. \quad (80)$$

The stability is determined by the second derivative. A direct differentiation gives

$$\left. \frac{d^2 V_{\text{eff}}}{dr^2} \right|_{\text{circ}} = \frac{2c^2 r_s (r - 3r_s)}{r^3 (2r - 3r_s)}. \quad (81)$$

Therefore stable circular orbits require

$$r > 3r_s. \quad (82)$$

The innermost stable circular orbit is

$$r_{\text{ISCO}} = 3r_s = \frac{6GM}{c^2}. \quad (83)$$

At this radius,

$$\frac{E_{\text{ISCO}}}{c^2} = \sqrt{\frac{8}{9}}, \quad (84)$$

so the maximum binding efficiency for a thin disk around a nonrotating black hole is

$$\eta = 1 - \sqrt{\frac{8}{9}} \approx 0.0572. \quad (85)$$

This result is fundamental in accretion theory [12, 13].

The marginally bound orbit is defined by $E = c^2$. Applying this condition to (80),

$$1 = \frac{(r - r_s)^2}{r(r - \frac{3}{2}r_s)}. \quad (86)$$

Expanding and simplifying gives

$$r_{\text{mb}} = 2r_s = \frac{4GM}{c^2}. \quad (87)$$

Circular orbits with $3r_s/2 < r < 3r_s$ exist but are unstable. The marginally bound orbit lies in this unstable range.

The angular frequency measured with respect to Schwarzschild coordinate time is obtained from $\Omega = \dot{\phi}/\dot{t}$. Substituting (66), (67), (78), and (80) gives the remarkably Newtonian-looking result

$$\Omega^2 = \left(\frac{d\phi}{dt}\right)^2 = \frac{GM}{r^3}. \quad (88)$$

The interpretation is not Newtonian, because t is the time of an observer at infinity and r is an areal radius. A local static observer at radius r measures the azimuthal speed

$$v_{\text{loc}} = \frac{r\Omega}{\sqrt{1 - r_s/r}} = c\sqrt{\frac{r_s}{2(r - r_s)}}. \quad (89)$$

At the ISCO, this gives

$$v_{\text{loc}}(r_{\text{ISCO}}) = \frac{c}{2}. \quad (90)$$

Small radial oscillations about a stable circular orbit have coordinate epicyclic frequency

$$\kappa^2 = \Omega^2 \left(1 - \frac{3r_s}{r}\right). \quad (91)$$

Thus κ vanishes at the ISCO, which provides an equivalent dynamical definition of marginal stability.

6.3. Photon sphere and shadow radius

For photons, circular null orbits satisfy

$$\frac{d}{dr} \left[\left(1 - \frac{r_s}{r}\right) \frac{1}{r^2} \right] = 0. \quad (92)$$

The derivative is

$$-\frac{2}{r^3} + \frac{3r_s}{r^4} = 0. \quad (93)$$

Thus the photon sphere is located at

$$r_{\text{ph}} = \frac{3}{2}r_s = \frac{3GM}{c^2}. \quad (94)$$

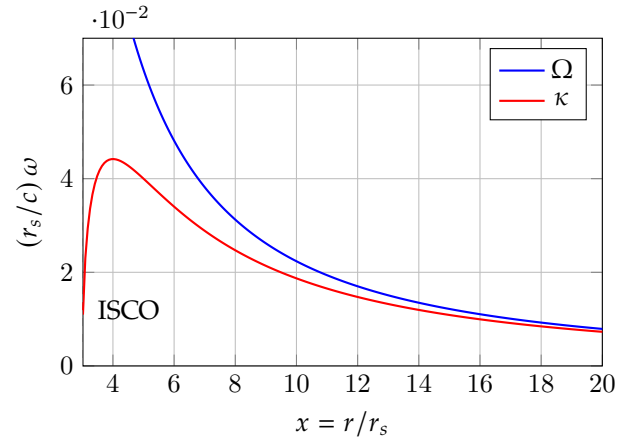


Figure 4. Coordinate orbital frequency Ω and radial epicyclic frequency κ for circular Schwarzschild orbits, expressed in dimensionless units. The epicyclic frequency tends to zero at $r = 3r_s$, signaling the loss of stable circular motion.

The photon impact parameter is

$$b = \frac{cL_\gamma}{E_\gamma}. \quad (95)$$

At a radial turning point,

$$\frac{E_\gamma^2}{c^2} = \left(1 - \frac{r_s}{r}\right) \frac{L_\gamma^2}{r^2}, \quad (96)$$

so

$$b^2 = \frac{r^2}{1 - r_s/r}. \quad (97)$$

At the photon sphere,

$$b_{\text{crit}} = \frac{(3/2)r_s}{\sqrt{1/3}} = \frac{3\sqrt{3}}{2}r_s. \quad (98)$$

Using $r_s = 2GM/c^2$,

$$b_{\text{crit}} = \sqrt{27} \frac{GM}{c^2}. \quad (99)$$

This is the radius of the Schwarzschild black-hole shadow as seen by a distant observer in asymptotically flat spacetime. The Event Horizon Telescope image of M87* measured an emission ring with angular diameter of order $40 \mu\text{as}$, consistent with the strong-lensing scale set by (99) for a mass near $6.5 \times 10^9 M_\odot$ [15, 16].

7. Radial Infall and the Event Horizon

7.1. Proper time for infall from rest at infinity

For radial infall, $L = 0$. Consider a massive particle released from rest at infinity. Its conserved energy is

$$E = c^2. \quad (100)$$

Equation (69) becomes

$$\dot{r}^2 = c^2 - c^2 \left(1 - \frac{r_s}{r}\right) = \frac{c^2 r_s}{r}. \quad (101)$$

For inward motion,

$$\frac{dr}{d\tau} = -c\sqrt{\frac{r_s}{r}}. \quad (102)$$

Integrating from r_0 to r ,

$$\tau(r) - \tau(r_0) = \frac{1}{c\sqrt{r_s}} \int_r^{r_0} \sqrt{r'} dr'. \quad (103)$$

Therefore

$$\tau(r) - \tau(r_0) = \frac{2}{3c\sqrt{r_s}} \left(r_0^{3/2} - r^{3/2} \right). \quad (104)$$

The proper time to reach the horizon is finite:

$$\Delta\tau_{r_0 \rightarrow r_s} = \frac{2r_s}{3c} \left[\left(\frac{r_0}{r_s} \right)^{3/2} - 1 \right]. \quad (105)$$

7.2. Schwarzschild coordinate time

The conserved energy equation gives

$$\frac{dt}{d\tau} = \left(1 - \frac{r_s}{r} \right)^{-1}. \quad (106)$$

Combining with (102),

$$\frac{dt}{dr} = -\frac{1}{c} \frac{\sqrt{r/r_s}}{1 - r_s/r}. \quad (107)$$

Let

$$x = \frac{r}{r_s}. \quad (108)$$

Then

$$\frac{c}{r_s} \frac{dt}{dx} = -\frac{x^{3/2}}{x-1}. \quad (109)$$

With $u = \sqrt{x}$,

$$\frac{x^{3/2}}{x-1} dx = \frac{2u^4}{u^2-1} du = 2 \left(u^2 + 1 + \frac{1}{u^2-1} \right) du. \quad (110)$$

An antiderivative is

$$F(x) = \frac{2}{3}x^{3/2} + 2x^{1/2} + \ln \left| \frac{\sqrt{x}-1}{\sqrt{x}+1} \right|. \quad (111)$$

Thus

$$\frac{c}{r_s} [t(r) - t(r_0)] = F\left(\frac{r_0}{r_s}\right) - F\left(\frac{r}{r_s}\right). \quad (112)$$

As $r \rightarrow r_s^+$, the logarithm in F tends to $-\infty$, and hence

$$t(r) \rightarrow +\infty. \quad (113)$$

The infalling observer crosses the horizon in finite proper time, but Schwarzschild coordinate time assigns the crossing to $t = \infty$.

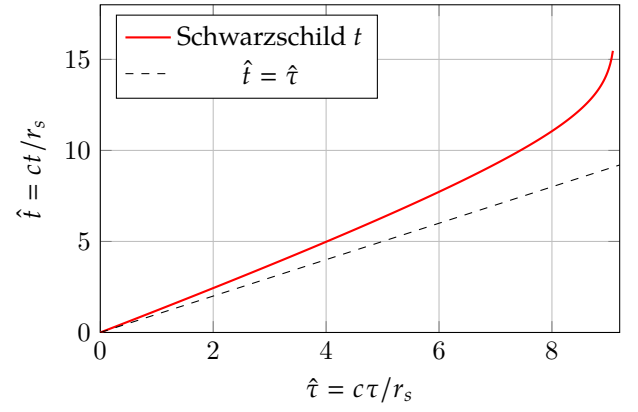


Figure 5. Coordinate time versus proper time for radial infall from rest at infinity, plotted for the segment from $r_0 = 6r_s$ to $r = 1.05r_s$. The proper time remains finite, while Schwarzschild coordinate time grows logarithmically near the horizon.

7.3. Coordinate singularity and Eddington–Finkelstein coordinates

The divergence of g_{rr} at $r = r_s$ in (53) is not a curvature singularity. For example, the Kretschmann scalar is

$$R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = \frac{12r_s^2}{r^6} = \frac{48G^2M^2}{c^4r^6}, \quad (114)$$

which is finite at $r = r_s$ and diverges only at $r = 0$.

Define the tortoise coordinate

$$r_* = r + r_s \ln \left| \frac{r}{r_s} - 1 \right|. \quad (115)$$

The ingoing Eddington–Finkelstein time coordinate is

$$v = t + \frac{r_*}{c}. \quad (116)$$

Since

$$c dt = c dv - \frac{dr}{1 - r_s/r}, \quad (117)$$

substitution into (53) yields

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) c^2 dv^2 + 2c dv dr + r^2 d\Omega^2. \quad (118)$$

All metric components in (118) are finite at $r = r_s$. The horizon is therefore a regular null hypersurface for infalling observers, not a physical singularity.

8. Relativistic Tests in the Weak Field

8.1. Perihelion precession

For timelike geodesics, introduce $u = 1/r$. Combining the radial equation with $\dot{\phi} = L/r^2$ yields the relativistic orbit equation

$$\frac{d^2u}{d\phi^2} + u = \frac{GM}{L^2} + \frac{3GM}{c^2} u^2. \quad (119)$$

The first two terms are the Newtonian Kepler problem; the last term is the leading Schwarzschild correction. For a nearly Keplerian ellipse,

$$u_0(\phi) = \frac{GM}{L^2} [1 + e \cos \phi], \quad (120)$$

where e is eccentricity. Perturbation theory gives an advance of perihelion per orbit

$$\Delta\phi = \frac{6\pi GM}{c^2 a(1 - e^2)}, \quad (121)$$

where a is the semimajor axis. This is one of the classic tests of general relativity.

For Mercury,

$$a = 5.7909 \times 10^{10} \text{ m}, \quad e = 0.2056, \quad GM_\odot = 1.3271 \times 10^{20} \text{ m}^3 \text{ s}^{-2} \quad (122)$$

Equation (121) gives

$$\Delta\phi = 5.02 \times 10^{-7} \text{ rad orbit}^{-1} = 0.1035 \text{ arcsec orbit}^{-1}. \quad (123)$$

Mercury completes about 415 orbits per century, hence

$$\Delta\phi_{\text{century}} \approx 43.0 \text{ arcsec century}^{-1}. \quad (124)$$

This accounts for the anomalous residual precession known before Einstein's theory [2, 6].

8.2. Light deflection and time delay

For null geodesics in the weak field, the leading deflection angle of a ray with impact parameter b is

$$\alpha = \frac{4GM}{c^2 b}. \quad (125)$$

The factor of four is twice the value suggested by a naive Newtonian corpuscular calculation: half arises from the temporal curvature and half from spatial curvature. For a ray grazing the solar limb, $b = R_\odot = 6.957 \times 10^8 \text{ m}$, so

$$\alpha_\odot = 8.49 \times 10^{-6} \text{ rad} = 1.75 \text{ arcsec}. \quad (126)$$

This was the effect measured during the 1919 solar eclipse expedition led by Eddington and collaborators [4].

The same weak-field null geometry predicts the Shapiro radar time delay [11]. For a signal passing near a mass with closest approach b and endpoints at radii r_1 and r_2 , the leading excess round-trip-independent one-way delay is

$$\Delta t_{\text{Shapiro}} = \frac{2GM}{c^3} \ln \left(\frac{4r_1 r_2}{b^2} \right). \quad (127)$$

The logarithm is a signature of integrating the weak Schwarzschild potential along an almost straight light path. For superior conjunction experiments involving Earth and an inner planet, the effect is of order hundreds of microseconds, far larger than modern timing precision.

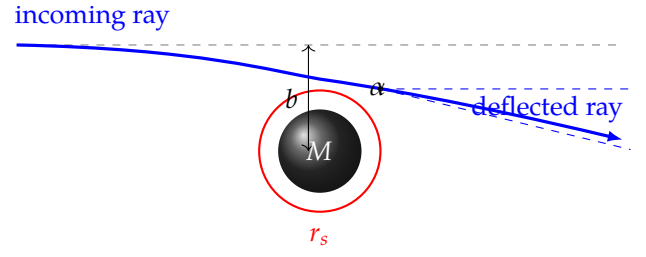


Figure 6. Weak-field light bending by a compact mass. A photon passing with impact parameter b is deflected through $\alpha = 4GM/(c^2 b)$ to leading order.

9. Worked Numerical Applications

It is useful to collect characteristic Schwarzschild scales. For one solar mass,

$$\frac{GM_\odot}{c^2} = 1.4766 \text{ km}, \quad r_s(M_\odot) = 2.9533 \text{ km}. \quad (128)$$

The ISCO radius for a Schwarzschild black hole of mass M is

$$r_{\text{ISCO}} = 8.8599 \left(\frac{M}{M_\odot} \right) \text{ km}. \quad (129)$$

The photon-sphere radius is

$$r_{\text{ph}} = 4.4299 \left(\frac{M}{M_\odot} \right) \text{ km}. \quad (130)$$

The shadow radius is

$$b_{\text{crit}} = 7.672 \left(\frac{M}{M_\odot} \right) \text{ km}. \quad (131)$$

For Sgr A*, taking

$$M_{\text{SgrA}^*} = 4.3 \times 10^6 M_\odot, \quad (132)$$

one finds

$$r_s = 1.27 \times 10^7 \text{ km} = 0.085 \text{ AU}. \quad (133)$$

The Schwarzschild shadow diameter is approximately

$$2b_{\text{crit}} = 6.60 \times 10^7 \text{ km} = 0.441 \text{ AU}. \quad (134)$$

At a distance $D \approx 8.2 \text{ kpc}$, this corresponds to an angular diameter

$$\theta_{\text{sh}} \approx \frac{2b_{\text{crit}}}{D} = 53.8 \mu\text{as}. \quad (135)$$

This is of the order observed by horizon-scale very-long-baseline interferometry.

For M87*, use

$$M_{\text{M87}^*} = 6.5 \times 10^9 M_\odot, \quad D = 16.8 \text{ Mpc}. \quad (136)$$

Then

$$b_{\text{crit}} = 4.99 \times 10^{10} \text{ km} = 333 \text{ AU}. \quad (137)$$

The angular diameter is

$$\theta_{\text{diam}} \approx \frac{2b_{\text{crit}}}{D} = 39.7 \mu\text{as}, \quad (138)$$

Table 1. Characteristic Schwarzschild radii for selected masses. Values scale linearly with M .

Object	r_s	r_{ph}	r_{ISCO}
$1M_\odot$	2.95 km	4.43 km	8.86 km
$10M_\odot$	29.5 km	44.3 km	88.6 km
Sgr A*	1.27×10^7 km	1.90×10^7 km	3.81×10^7 km
M87*	1.92×10^{10} km	2.88×10^{10} km	5.76×10^{10} km

consistent with the EHT ring diameter of order 40–45 μas when spin, emission geometry, plasma physics, and lensing transfer are included [15, 16].

As a final numerical illustration, consider a clock at radius $r = 10r_s$. From (55),

$$\frac{d\tau}{dt} = \sqrt{1 - \frac{1}{10}} = 0.9487. \quad (139)$$

A static clock at $10r_s$ runs about 5.13% slower than a clock at infinity. At $r = 2r_s$,

$$\frac{d\tau}{dt} = \frac{1}{\sqrt{2}} = 0.7071, \quad (140)$$

and the gravitational redshift to infinity is

$$z = \sqrt{2} - 1 = 0.414. \quad (141)$$

9.1. Additional worked example: a neutron-star exterior

A useful application of the same exterior metric is a non-rotating neutron star. Take

$$M = 1.4M_\odot, \quad R = 12 \text{ km}. \quad (142)$$

The exterior Schwarzschild radius is

$$r_s = 2.9533(1.4) \text{ km} = 4.13 \text{ km}. \quad (143)$$

The compactness is therefore

$$\frac{GM}{Rc^2} = \frac{r_s}{2R} = 0.172. \quad (144)$$

A photon emitted from the stellar surface and received at infinity has redshift

$$z = \left(1 - \frac{4.13}{12}\right)^{-1/2} - 1 = 0.235. \quad (145)$$

Thus an atomic line emitted at rest wavelength λ_e would be detected at

$$\lambda_\infty = (1 + z)\lambda_e = 1.235\lambda_e. \quad (146)$$

For comparison, the surface proper acceleration from (63) is

$$a_R = \frac{GM}{R^2\sqrt{1 - r_s/R}} = 1.59 \times 10^{12} \text{ m s}^{-2}. \quad (147)$$

This is about 1.6×10^{11} times terrestrial gravity. The calculation illustrates why compact-star spectra, when

identifiable, probe relativistic gravity even though the object is not a black hole.

The same parameters give an exterior circular-orbit frequency at the stellar radius,

$$f_{\text{orb}} = \frac{1}{2\pi} \sqrt{\frac{GM}{R^3}} = 1.65 \times 10^3 \text{ Hz}. \quad (148)$$

Because $R > r_{\text{ISCO}} = 6GM/c^2 = 12.4 \text{ km}$ is only marginally false for these particular rounded numbers, modest changes in mass or radius determine whether the innermost disk is terminated by the stellar surface or by the relativistic ISCO. This is one reason kilohertz variability in X-ray binaries is sensitive to the compactness of neutron stars [10].

9.2. Additional worked example: solar Shapiro delay

For a radar signal sent from Earth to a spacecraft on the far side of the Sun, let

$$r_1 = 1.00 \text{ AU}, \quad r_2 = 1.50 \text{ AU}, \quad b = R_\odot. \quad (149)$$

Using $GM_\odot/c^3 = 4.9255 \mu\text{s}$ and (127),

$$\ln\left(\frac{4r_1r_2}{b^2}\right) = \ln\left[\frac{4(1.496 \times 10^{11})(2.244 \times 10^{11})}{(6.957 \times 10^8)^2}\right] \quad (150)$$

$$= 12.53. \quad (151)$$

The one-way delay is then

$$\Delta t_{\text{Shapiro}} = 2(4.9255 \mu\text{s})(12.53) = 123 \mu\text{s}. \quad (152)$$

A two-way radar experiment doubles this leading contribution. The measurement is not a strong-field test in the black-hole sense, but it is a sensitive probe of the spatial curvature encoded in the Schwarzschild metric.

10. Extensions, Generalisations, and Observational Diagnostics

The Schwarzschild metric is the exterior solution for a neutral, nonrotating, asymptotically flat object in vacuum. Relaxing any of these descriptors leads to useful generalisations. With a cosmological constant, the vacuum field equations become

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0, \quad (153)$$

and the spherically symmetric solution is Schwarzschild–de Sitter or Schwarzschild–anti-de Sitter:

$$ds^2 = -f_\Lambda(r)c^2dt^2 + f_\Lambda(r)^{-1}dr^2 + r^2d\Omega^2, \quad (154)$$

where

$$f_{\Lambda}(r) = 1 - \frac{r_s}{r} - \frac{\Lambda r^2}{3}. \quad (155)$$

For astrophysical black holes, Λr^2 is normally negligible near the horizon, but the term matters conceptually because it introduces a cosmological horizon when $\Lambda > 0$.

If the central body carries electric charge Q , the exterior electrovacuum geometry is Reissner–Nordstrom rather than Schwarzschild:

$$f_Q(r) = 1 - \frac{2GM}{c^2 r} + \frac{GQ^2}{4\pi\epsilon_0 c^4 r^2}. \quad (156)$$

Astrophysical black holes are expected to be nearly neutral because surrounding plasma efficiently discharges large electric fields, but (156) is important theoretically as the simplest example with inner and outer horizons. Rotation is astrophysically much more important. In the Kerr spacetime, the single Schwarzschild photon sphere is replaced by a photon region, the ISCO depends on the sense of orbital motion, and frame dragging couples temporal and azimuthal motion [7].

Precision tests often describe weak-field deviations from general relativity using the parametrized post-Newtonian parameters β and γ [14]. In isotropic weak-field form,

$$g_{tt} = -c^2 \left[1 - \frac{2GM}{c^2 r} + 2\beta \left(\frac{GM}{c^2 r} \right)^2 \right], \quad (157)$$

$$g_{rr} = 1 + 2\gamma \frac{GM}{c^2 r}. \quad (158)$$

General relativity predicts

$$\beta = 1, \quad \gamma = 1. \quad (159)$$

The leading light-deflection angle becomes

$$\alpha_{\text{PPN}} = (1 + \gamma) \frac{2GM}{c^2 b}, \quad (160)$$

while the perihelion advance becomes

$$\Delta\phi_{\text{PPN}} = \frac{6\pi GM}{c^2 a(1 - e^2)} \left(\frac{2 + 2\gamma - \beta}{3} \right). \quad (161)$$

The Schwarzschild results are recovered by inserting (159). These formulae show why light propagation and orbital precession test complementary combinations of metric coefficients.

Strong-field observations are less easily summarized by one parameter, but the Schwarzschild solution still supplies useful diagnostics. First, the shadow angular diameter scales linearly with M/D through (99). Second, thermal thin-disk spectra depend on the ISCO through the efficiency (85). Third, high-frequency quasi-periodic phenomena, if associated with orbital or epicyclic motion, scale approximately as M^{-1} through (88) and (91). Finally, gravitational waves from compact binaries probe the Schwarzschild geometry in the test-mass limit and the nonlinear two-body problem more generally; the late inspiral and ringdown frequencies are set by the same relativistic length scale GM/c^2 .

11. Discussion

The derivation above rests on several assumptions. First, the spacetime is vacuum outside the gravitating object. This is accurate for the exterior of a star or black hole when surrounding gas, radiation, and magnetic fields contribute negligibly to the stress-energy. In accreting systems the Schwarzschild metric is usually a fixed background approximation, while plasma dynamics and radiation transfer are treated separately. Second, the solution is nonrotating. Real astrophysical black holes are expected to rotate because angular momentum is ubiquitous in gravitational collapse and accretion. Their exterior geometry is described by the Kerr solution, whose horizon radius, ISCO, photon region, and shadow depend on spin and observer inclination [7, 5]. Nevertheless, Schwarzschild remains the exact zero-spin limit and the clearest model of horizon formation.

Third, the coordinates used in (53) are not globally regular. The divergence of g_{rr} at $r = r_s$ is a coordinate singularity analogous to a polar-coordinate singularity, not a curvature singularity. Eddington–Finkelstein coordinates demonstrate that infalling light rays and particles cross the horizon smoothly. The true invariant singularity is at $r = 0$, where (114) diverges and classical general relativity predicts geodesic incompleteness.

The effective potential formalism gives direct physical insight. In Newtonian gravity, stable circular orbits exist at all radii for a $1/r$ potential. In Schwarzschild spacetime, the factor $1 - r_s/r$ multiplying the angular barrier causes the potential to develop an inner maximum and eventually lose its minimum. This is why the ISCO exists. The ISCO is not a material surface; rather, it is a dynamical transition radius. In thin-disk theory, gas spirals slowly inward outside the ISCO through viscous angular-momentum transport and then plunges rapidly inside it [12, 13]. The binding energy at the ISCO gives the canonical Schwarzschild accretion efficiency of 5.7%.

Photon dynamics is even more strongly relativistic. The photon sphere at $r = 1.5r_s$ is outside the event horizon but inside the ISCO. It is unstable: a small outward perturbation allows escape, while a small inward perturbation leads to capture. The critical impact parameter $b_{\text{crit}} = \sqrt{27}GM/c^2$ defines the idealized shadow boundary for a Schwarzschild black hole illuminated by optically thin emission. Actual images depend on plasma emissivity, absorption, Doppler boosting, gravitational redshift, spin, turbulence, and instrumental response. The EHT observations of M87* and Sgr A* therefore do not show the event horizon directly as a material edge; rather, they reveal a lensed emission morphology whose characteristic angular scale is set by the photon capture region [15, 16].

The weak-field predictions discussed here remain among the most important empirical successes of general relativity. Mercury's perihelion advance tests the timelike geodesic structure of the Schwarzschild field.

Solar light deflection tests null geodesics and spatial curvature. Gravitational redshift tests the metric time component. Modern measurements, including radar time delay, binary pulsars, stellar orbits around Sgr A*, gravitational-wave detections, and black-hole imaging, extend these tests into increasingly strong-field regimes [11, 14, 15].

Finally, the uniqueness implied by Birkhoff's theorem should not be overextended. It applies to spherically symmetric vacuum regions. It does not say that all compact objects are Schwarzschild, nor does it apply in nonspherical collapse, in the presence of matter fields, or in modified theories of gravity. It is nevertheless the reason why the exterior of a static spherical star and that of a nonrotating black hole share the same local metric outside the stellar surface or horizon.

12. Conclusion

Starting from the static spherically symmetric ansatz with two unknown functions $A(r)$ and $B(r)$, we explicitly computed the Christoffel symbols and Ricci tensor components and solved the vacuum Einstein equations. The equations first imply $A(r)B(r) = 1$ after imposing asymptotic flatness. The angular equation then gives $A(r) = 1 + C/r$, and the Newtonian weak-field limit fixes $C = -2GM/c^2$. The resulting Schwarzschild metric is the unique asymptotically flat spherically symmetric vacuum solution.

The geodesic analysis reveals the physical content of the metric. Massive particles move in the effective potential

$$V_{\text{eff}} = (1 - r_s/r)(c^2 + L^2/r^2), \quad (162)$$

which yields stable circular orbits only for $r > 3r_s$, the ISCO at $r = 3r_s = 6GM/c^2$, and the marginally bound orbit at $r = 2r_s$. Photons possess an unstable circular orbit at $r = 1.5r_s$, producing a shadow radius $\sqrt{27}GM/c^2$. Radial infall shows that the horizon is reached in finite proper time but infinite Schwarzschild coordinate time, while Eddington–Finkelstein coordinates display the regularity of the horizon. The same geometry accounts quantitatively for gravitational redshift, Mercury's anomalous perihelion precession, solar light deflection, Shapiro delay, compact-star surface redshifts, and the characteristic angular scale of horizon-resolving observations such as the EHT image of M87*.

A. Algebraic Details for Orbit Equations

This appendix records intermediate algebra leading from the radial first integral to the orbit equation used in the weak-field precession calculation. For timelike motion, write $u = 1/r$ and use

$$\dot{r} = \frac{dr}{d\phi} \dot{\phi} = -L \frac{du}{d\phi}. \quad (163)$$

Substitution into (69) gives

$$L^2 \left(\frac{du}{d\phi} \right)^2 + (1 - r_s u)(c^2 + L^2 u^2) = \frac{E^2}{c^2}. \quad (164)$$

Expanding the potential terms,

$$L^2 u'^2 + c^2 - c^2 r_s u + L^2 u^2 - r_s L^2 u^3 = \frac{E^2}{c^2}, \quad (165)$$

where $u' = du/d\phi$. Differentiating with respect to ϕ and dividing by $2L^2 u'$ for noncircular segments yields

$$\frac{d^2 u}{d\phi^2} + u = \frac{c^2 r_s}{2L^2} + \frac{3r_s}{2} u^2. \quad (166)$$

Since $r_s = 2GM/c^2$, this becomes

$$\frac{d^2 u}{d\phi^2} + u = \frac{GM}{L^2} + \frac{3GM}{c^2} u^2, \quad (167)$$

which is (119).

For null motion, the analogous calculation begins with

$$L_\gamma^2 \left(\frac{du}{d\phi} \right)^2 + (1 - r_s u)L_\gamma^2 u^2 = \frac{E_\gamma^2}{c^2}. \quad (168)$$

After division by L_γ^2 and differentiation,

$$\frac{d^2 u}{d\phi^2} + u = \frac{3r_s}{2} u^2 = \frac{3GM}{c^2} u^2. \quad (169)$$

In the weak-field limit the right-hand side may be evaluated on the zeroth-order straight-line trajectory, giving the leading light deflection (125).

The epicyclic result (91) follows from expanding the radial equation about a circular orbit $r = r_0 + \delta r$. With $V'_{\text{eff}}(r_0) = 0$,

$$\frac{d^2 \delta r}{d\tau^2} + \frac{1}{2} V''_{\text{eff}}(r_0) \delta r = 0. \quad (170)$$

The proper-time radial frequency is therefore

$$\omega_r^2 = \frac{1}{2} V''_{\text{eff}}(r_0) = \frac{c^2 r_s (r_0 - 3r_s)}{r_0^3 (2r_0 - 3r_s)}. \quad (171)$$

Changing from proper time to coordinate time with the circular-orbit value of $dt/d\tau$ gives

$$\left(\frac{dt}{d\tau} \right)^2 = \frac{r_0}{r_0 - \frac{3}{2}r_s}. \quad (172)$$

Thus

$$\kappa^2 = \omega_r^2 \left(\frac{d\tau}{dt} \right)^2 = \frac{GM}{r_0^3} \left(1 - \frac{3r_s}{r_0} \right), \quad (173)$$

as quoted in (91).

References

- [1] K. Schwarzschild, "Über das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie," Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften, 189–196 (1916).
- [2] A. Einstein, "Die Grundlage der allgemeinen Relativitätstheorie," Annalen der Physik **49**, 769–822 (1916).
- [3] G. D. Birkhoff, *Relativity and Modern Physics*, Harvard University Press, Cambridge (1923).
- [4] A. S. Eddington, *Space, Time and Gravitation*, Cambridge University Press, Cambridge (1920).
- [5] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation*, W. H. Freeman, San Francisco (1973).
- [6] S. Weinberg, *Gravitation and Cosmology*, Wiley, New York (1972).
- [7] S. Chandrasekhar, *The Mathematical Theory of Black Holes*, Oxford University Press, Oxford (1983).
- [8] R. M. Wald, *General Relativity*, University of Chicago Press, Chicago (1984).
- [9] S. M. Carroll, *Spacetime and Geometry: An Introduction to General Relativity*, Addison-Wesley, San Francisco (2004).
- [10] B. W. Carroll and D. A. Ostlie, *An Introduction to Modern Astrophysics*, 2nd ed., Cambridge University Press, Cambridge (2017).
- [11] I. I. Shapiro, "Fourth test of general relativity," Physical Review Letters **13**, 789–791 (1964).
- [12] N. I. Shakura and R. A. Sunyaev, "Black holes in binary systems. Observational appearance," Astronomy and Astrophysics **24**, 337–355 (1973).
- [13] I. D. Novikov and K. S. Thorne, "Astrophysics of black holes," in *Black Holes*, edited by C. DeWitt and B. S. DeWitt, Gordon and Breach, New York, 343–450 (1973).
- [14] C. M. Will, "The confrontation between general relativity and experiment," Living Reviews in Relativity **17**, 4 (2014).
- [15] Event Horizon Telescope Collaboration, "First M87 Event Horizon Telescope Results. I. The shadow of the supermassive black hole," Astrophysical Journal Letters **875**, L1 (2019).
- [16] Event Horizon Telescope Collaboration, "First M87 Event Horizon Telescope Results. VI. The shadow and mass of the central black hole," Astrophysical Journal Letters **875**, L6 (2019).