



GALACTIC DYNAMICS

The Jeans Equations, Galactic Rotation Curves and the NFW Dark Matter Potential Profile

Academic Research Paper

Theoretical Astrophysics — Step-by-Step Analytic Derivations, Vector Diagrams and Numerical Tables

The Jeans Equations, Galactic Rotation Curves and the NFW Dark Matter Potential Profile

Galactic Dynamics

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Abstract. Galactic dynamics relates the observed motions of stars and gas to the gravitational field generated by luminous and dark matter. This paper gives a self-contained derivation of the collisionless Boltzmann equation, its velocity moments, and the Jeans equations, emphasizing the assumptions under which phase-space dynamics becomes a practical mass estimator. We derive the spherical Jeans equation with velocity anisotropy, the cylindrical Jeans equation and asymmetric drift, and the local vertical Jeans equation used to infer the dark-matter density near the Sun. We then connect these dynamical tools to galactic rotation curves, showing why a point mass gives a Keplerian decline, why a singular isothermal sphere gives an exactly flat curve, and why the observed curves of spiral galaxies motivated dark halos. The Navarro–Frenk–White profile is treated in detail: enclosed mass, gravitational potential, circular-speed curve, maximum velocity, logarithmic slopes, concentration, virial mass, and the characteristic-density–concentration relation are all derived explicitly. Worked numerical applications include Milky-Way-like halos, dwarf-spheroidal mass estimates, asymmetric drift, and the local dark-matter density. We close with limitations of Jeans modelling, the cusp–core problem, and the observational status of cold-dark-matter halo profiles.

Keywords: galactic dynamics; Jeans equations; rotation curves; dark matter halos; NFW profile; dwarf spheroidals; local dark matter density

1. Introduction

The central inference problem of galactic dynamics is deceptively simple: given positions and velocities of tracers, determine the gravitational potential and therefore the mass distribution. The difficulty is that stars in galaxies are not collisional gas particles. Their mean free paths are far larger than the system size, their two-body relaxation times are normally longer than the Hubble time, and their motions preserve phase-space structure rather than approaching local thermodynamic equilibrium. Galactic dynamics is therefore built on the collisionless Boltzmann equation, also called the Vlasov equation, and on its velocity moments, the Jeans equations.

Historically, Jeans’ analysis of stellar motions [1] and Eddington’s distribution-function methods [2] established that stellar systems can be treated statistically without being collisional fluids. Chandrasekhar’s theory of stellar encounters [3] clarified when the collisionless approximation is valid. Modern syntheses are given in [4, 5]. In the second half of the twentieth century, galactic rotation curves transformed these formal tools into evidence for non-luminous matter. Rubin, Ford, and collaborators found nearly flat rotation curves in spiral galaxies [6], and Bosma showed that extended neutral-hydrogen disks trace the same phenomenon well beyond the optical radius [7]. If most mass followed the starlight, circular speeds should decline approximately as $r^{-1/2}$ at large radii; instead, many galaxies show $v_c(r) \simeq \text{constant}$ over tens of kiloparsecs.

Cold dark matter cosmological simulations made this problem predictive. Navarro, Frenk, and White found that simulated halos are well described by a two-

parameter density profile,

$$\rho(r) = \frac{\rho_s}{(r/r_s)(1 + r/r_s)^2}, \quad (1)$$

with inner logarithmic slope -1 and outer slope -3 [8, 9]. Equation (1) is not merely a convenient fit; it links galactic kinematics to cosmological structure formation through the virial mass and concentration. Yet the same profile generates an enduring tension: many dwarf and low-surface-brightness galaxies appear better fitted by constant-density cores, such as the Burkert profile [10], than by cusps. This is the cusp–core problem.

The purpose of this paper is pedagogical but rigorous. We derive the kinetic equations from phase-space continuity, obtain the Jeans equations step by step, and apply them to rotation curves, dwarf spheroidals, disk kinematics, and NFW halos. We emphasize the physical content of each equation: which stress supports a stellar system, how anisotropy enters a mass estimate, why pressure support makes stellar rotation slower than circular rotation, and how the shape of a halo density profile maps into a circular-speed curve.

2. Physical and Mathematical Framework

2.1. Phase space and distribution functions

Definition 1. The one-particle distribution function $f(\mathbf{x}, \mathbf{v}, t)$ is defined so that the number of stars in a phase-space volume element is

$$dN = f(\mathbf{x}, \mathbf{v}, t) d^3x d^3v. \quad (2)$$

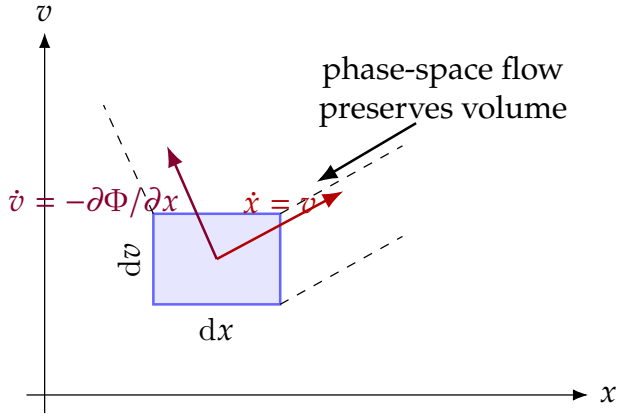


Figure 1. A phase-space element advected by Hamiltonian flow. Conservation of tracer number in this moving volume gives the collisionless Boltzmann equation.

Its zeroth velocity moment gives the number density,

$$v(x, t) = \int f(x, v, t) d^3v. \quad (3)$$

If all tracers have identical mass $m_\star$, the tracer mass density is $\rho_\star = m_\star v$. In Jeans modelling, however, the tracer density v need not equal the gravitating density ρ . Dwarf spheroidal galaxies provide the cleanest example: visible stars trace the potential, but the dominant mass is dark.

The mean of any velocity function $q(v)$ is

$$v\langle q \rangle = \int q(v) f(x, v, t) d^3v. \quad (4)$$

In particular,

$$\bar{v}_i = \langle v_i \rangle, \quad \sigma_{ij}^2 = \langle (v_i - \bar{v}_i)(v_j - \bar{v}_j) \rangle. \quad (5)$$

The tensor $v\sigma_{ij}^2$ is the stellar analogue of pressure, but it is generally anisotropic.

2.2. Gravity and circular speed

The gravitational potential obeys Poisson's equation,

$$\nabla^2 \Phi = 4\pi G \rho, \quad (6)$$

where ρ is the total gravitating density. In spherical symmetry,

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\Phi}{dr} \right) = 4\pi G \rho(r). \quad (7)$$

The enclosed mass is

$$M(r) = 4\pi \int_0^r \rho(s) s^2 ds, \quad (8)$$

and integration of (7) gives

$$\frac{d\Phi}{dr} = \frac{GM(r)}{r^2}. \quad (9)$$

A circular orbit satisfies centrifugal balance,

$$\frac{v_c^2}{r} = \frac{d\Phi}{dr}, \quad (10)$$

so that

$$v_c^2(r) = \frac{GM(r)}{r}. \quad (11)$$

Thus a rotation curve is directly a mass profile only under circular motion and spherical symmetry; real disk galaxies require corrections for disk geometry, pressure support, bars, warps, and non-circular flows.

3. The Collisionless Boltzmann Equation

3.1. Derivation from phase-space continuity

Consider the six-dimensional phase-space coordinate

$$\mathbf{w} = (\mathbf{x}, \mathbf{v}). \quad (12)$$

The phase-space flow velocity is

$$\dot{\mathbf{w}} = (\dot{\mathbf{x}}, \dot{\mathbf{v}}) = (\mathbf{v}, -\nabla \Phi). \quad (13)$$

Conservation of the number of tracers in phase space requires

$$\frac{\partial f}{\partial t} + \nabla_{\mathbf{w}} \cdot (f \dot{\mathbf{w}}) = 0. \quad (14)$$

Expanding the six-dimensional divergence gives

$$\frac{\partial f}{\partial t} + \nabla_{\mathbf{x}} \cdot (f \mathbf{v}) + \nabla_{\mathbf{v}} \cdot [-f \nabla \Phi] = 0. \quad (15)$$

Since $\mathbf{v}$ is independent of $\mathbf{x}$ and $\nabla \Phi$ is independent of $\mathbf{v}$,

$$\nabla_{\mathbf{x}} \cdot (f \mathbf{v}) = \mathbf{v} \cdot \nabla_{\mathbf{x}} f, \quad (16)$$

$$\nabla_{\mathbf{v}} \cdot [-f \nabla \Phi] = -(\nabla \Phi) \cdot \nabla_{\mathbf{v}} f. \quad (17)$$

Therefore

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f - (\nabla \Phi) \cdot \nabla_{\mathbf{v}} f = 0. \quad (18)$$

In index notation this is

$$\frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial x_i} - \frac{\partial \Phi}{\partial x_i} \frac{\partial f}{\partial v_i} = 0, \quad (19)$$

with summation over repeated indices.

Theorem 1 (Collisionless Boltzmann equation). *For a stellar system in which close encounters may be neglected, the distribution function is conserved along phase-space characteristics:*

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \dot{\mathbf{x}} \cdot \nabla_{\mathbf{x}} f + \dot{\mathbf{v}} \cdot \nabla_{\mathbf{v}} f = 0. \quad (20)$$

The physical meaning is Liouville conservation: stars move through phase space without creation, destruction, or collisional diffusion. The only way f changes at a fixed point is by advection through that point.

The collisionless approximation is accurate when the two-body relaxation time is large. For a system of N comparable particles,

$$t_{\text{rel}} \simeq \frac{0.1N}{\ln N} t_{\text{cross}}, \quad (21)$$

where $t_{\text{cross}} \sim R/\sigma$. In a galaxy with $N \sim 10^{11}$, t_{rel} exceeds the age of the Universe. Globular clusters, with much smaller N , are less collisionless and require additional physics.

3.2. Steady states and Jeans theorem

If the potential is time independent, a steady distribution satisfies

$$\mathbf{v} \cdot \nabla_{\mathbf{x}} f - (\nabla \Phi) \cdot \nabla_{\mathbf{v}} f = 0. \quad (22)$$

Any function of isolating integrals of motion is a solution. For example, in a spherical potential the specific energy

$$E = \frac{1}{2}v^2 + \Phi(r) \quad (23)$$

and angular momentum

$$L^2 = r^2 v_t^2 \quad (24)$$

are integrals. A distribution $f(E, L)$ automatically satisfies (22). The Jeans equations use only low-order moments of f ; consequently they are less informative than a full distribution-function model, but they are much easier to apply to data.

4. Velocity Moments and the Jeans Equations

4.1. Zeroth moment: continuity equation

Integrate (19) over all velocities:

$$\int \left[\frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial x_i} - \frac{\partial \Phi}{\partial x_i} \frac{\partial f}{\partial v_i} \right] d^3v = 0. \quad (25)$$

The first term is

$$\int \frac{\partial f}{\partial t} d^3v = \frac{\partial v}{\partial t}. \quad (26)$$

The second term is

$$\int v_i \frac{\partial f}{\partial x_i} d^3v = \frac{\partial}{\partial x_i} \int v_i f d^3v = \frac{\partial (v \bar{v}_i)}{\partial x_i}. \quad (27)$$

The force term vanishes if $f \rightarrow 0$ sufficiently fast as $|v| \rightarrow \infty$:

$$-\frac{\partial \Phi}{\partial x_i} \int \frac{\partial f}{\partial v_i} d^3v = 0. \quad (28)$$

Hence

$$\frac{\partial v}{\partial t} + \frac{\partial (v \bar{v}_i)}{\partial x_i} = 0. \quad (29)$$

In vector notation,

$$\frac{\partial v}{\partial t} + \nabla \cdot (v \bar{\mathbf{v}}) = 0. \quad (30)$$

This is ordinary number conservation. It contains no gravitational force explicitly because gravity moves stars in velocity space but does not create or destroy them.

4.2. First moment: Cartesian Jeans equation

Multiply (19) by v_j and integrate:

$$\int v_j \left[\frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial x_i} - \frac{\partial \Phi}{\partial x_i} \frac{\partial f}{\partial v_i} \right] d^3v = 0. \quad (31)$$

The time term gives

$$\int v_j \frac{\partial f}{\partial t} d^3v = \frac{\partial (v \bar{v}_j)}{\partial t}. \quad (32)$$

The streaming term gives

$$\int v_j v_i \frac{\partial f}{\partial x_i} d^3v = \frac{\partial (v \langle v_i v_j \rangle)}{\partial x_i}. \quad (33)$$

For the force term integrate by parts:

$$\begin{aligned} -\frac{\partial \Phi}{\partial x_i} \int v_j \frac{\partial f}{\partial v_i} d^3v &= -\frac{\partial \Phi}{\partial x_i} [v_j f]_{\infty} \\ &\quad + \frac{\partial \Phi}{\partial x_i} \int f \frac{\partial v_j}{\partial v_i} d^3v \\ &= v \frac{\partial \Phi}{\partial x_j}. \end{aligned} \quad (34)$$

Thus

$$\frac{\partial (v \bar{v}_j)}{\partial t} + \frac{\partial (v \langle v_i v_j \rangle)}{\partial x_i} + v \frac{\partial \Phi}{\partial x_j} = 0. \quad (35)$$

Using

$$\langle v_i v_j \rangle = \sigma_{ij}^2 + \bar{v}_i \bar{v}_j, \quad (36)$$

and subtracting the continuity equation multiplied by $\bar{v}_j$, one obtains the Euler-like form

$$v \left(\frac{\partial \bar{v}_j}{\partial t} + \bar{v}_i \frac{\partial \bar{v}_j}{\partial x_i} \right) = -v \frac{\partial \Phi}{\partial x_j} - \frac{\partial (v \sigma_{ij}^2)}{\partial x_i}. \quad (37)$$

Equation (37) is the Jeans equation. It resembles the Euler equation, but the pressure is replaced by a stress tensor determined by the stellar orbit distribution. Because higher moments are not specified, the Jeans hierarchy is not closed without assumptions about anisotropy or distribution functions.

5. The Spherical Jeans Equation

5.1. Derivation with anisotropy

For a steady, non-rotating, spherically symmetric tracer population, $\bar{v}_r = \bar{v}_\theta = \bar{v}_\phi = 0$, and all quantities depend only on r . The radial Jeans equation is

$$\frac{d(v \sigma_r^2)}{dr} + \frac{v}{r} (2\sigma_r^2 - \sigma_\theta^2 - \sigma_\phi^2) = -v \frac{d\Phi}{dr}. \quad (38)$$

Spherical symmetry requires

$$\sigma_\theta^2 = \sigma_\phi^2. \quad (39)$$

Define the Binney anisotropy parameter

$$\beta(r) = 1 - \frac{\sigma_\theta^2 + \sigma_\phi^2}{2\sigma_r^2} = 1 - \frac{\sigma_t^2}{\sigma_r^2}, \quad (40)$$

where $\sigma_t^2 = \sigma_\theta^2 = \sigma_\phi^2$. Substituting into (38) gives

$$\frac{d(v\sigma_r^2)}{dr} + \frac{2\beta v\sigma_r^2}{r} = -v \frac{GM(r)}{r^2}. \quad (41)$$

Solving for mass,

$$M(r) = -\frac{r^2}{Gv} \left[\frac{d(v\sigma_r^2)}{dr} + \frac{2\beta v\sigma_r^2}{r} \right]. \quad (42)$$

Equivalently,

$$M(r) = -\frac{r\sigma_r^2}{G} \left[\frac{d \ln v}{d \ln r} + \frac{d \ln \sigma_r^2}{d \ln r} + 2\beta \right]. \quad (43)$$

The physical interpretation is transparent. Gravity pulls inward. It is balanced by the radial gradient of radial pressure, $d(v\sigma_r^2)/dr$, and by a curvature term governed by anisotropy. If $\beta > 0$, orbits are radially biased and less tangential pressure is available; for a fixed observed dispersion, the inferred mass changes.

For specified $M(r)$ and $\beta(r)$, (41) is a first-order linear equation. With integrating factor

$$I(r) = \exp \left[\int_r^r \frac{2\beta(s)}{s} ds \right], \quad (44)$$

the solution satisfying $v\sigma_r^2 \rightarrow 0$ at infinity is

$$v(r)\sigma_r^2(r) = \frac{1}{I(r)} \int_r^\infty I(s)v(s) \frac{GM(s)}{s^2} ds. \quad (45)$$

For constant β ,

$$I(r) = r^{2\beta}, \quad (46)$$

and therefore

$$v\sigma_r^2 = r^{-2\beta} \int_r^\infty s^{2\beta} v(s) \frac{GM(s)}{s^2} ds. \quad (47)$$

Remark 1. The observed line-of-sight dispersion is a projected quantity. If R is projected radius and $r^2 = R^2 + z^2$, then

$$\Sigma(R) = 2 \int_R^\infty \frac{v(r)r dr}{\sqrt{r^2 - R^2}}, \quad (48)$$

and

$$\Sigma\sigma_{\text{los}}^2 = 2 \int_R^\infty \left(1 - \beta \frac{R^2}{r^2} \right) \frac{v\sigma_r^2 r dr}{\sqrt{r^2 - R^2}}. \quad (49)$$

Equations (41) and (49) show the mass–anisotropy degeneracy: different pairs $M(r)$ and $\beta(r)$ can produce similar projected dispersions.

5.2. Application to dwarf spheroidal galaxies

Dwarf spheroidal galaxies have small baryonic masses, little gas, and large stellar velocity dispersions relative to their luminosities. A useful mass estimator follows from the spherical Jeans equation near the three-dimensional half-light radius $r_{1/2}$. At the radius where

$$\frac{d \ln v}{d \ln r} \simeq -3, \quad (50)$$

the enclosed mass is weakly sensitive to β [11]. The estimator is

$$M(r_{1/2}) \simeq \frac{3r_{1/2}\langle\sigma_{\text{los}}^2\rangle}{G}. \quad (51)$$

Since $r_{1/2} \simeq (4/3)R_e$ for many light profiles,

$$M(r_{1/2}) \simeq \frac{4R_e\langle\sigma_{\text{los}}^2\rangle}{G}. \quad (52)$$

For a typical classical dwarf with $R_e = 300$ pc and $\sigma_{\text{los}} = 10 \text{ km s}^{-1}$, using

$$G = 4.302 \times 10^{-3} \frac{\text{pc} (\text{km s}^{-1})^2}{M_\odot}, \quad (53)$$

we obtain

$$\begin{aligned} M(r_{1/2}) &= \frac{4(300 \text{ pc})(100 \text{ km}^2 \text{ s}^{-2})}{4.302 \times 10^{-3}} M_\odot \\ &= 2.79 \times 10^7 M_\odot. \end{aligned} \quad (54)$$

If the luminosity is $L = 5 \times 10^5 L_\odot$, then

$$\frac{M}{L} \simeq 56 \frac{M_\odot}{L_\odot}. \quad (55)$$

The high mass-to-light ratio is not a subtle modelling artifact; it follows from the virial scale $r\sigma^2/G$.

6. Axisymmetric Jeans Equations and Asymmetric Drift

6.1. Cylindrical radial Jeans equation

In cylindrical coordinates (R, ϕ, z) , for a steady axisymmetric disk with $\partial/\partial t = 0$ and $\partial/\partial \phi = 0$, the radial Jeans equation is

$$\frac{\partial(v\sigma_R^2)}{\partial R} + \frac{\partial(v\sigma_{Rz}^2)}{\partial z} + v \left(\frac{\sigma_R^2 - \overline{v_\phi^2}}{R} + \frac{\partial\Phi}{\partial R} \right) = 0. \quad (56)$$

The circular speed is

$$v_c^2(R, z) = R \frac{\partial\Phi}{\partial R}. \quad (57)$$

Writing

$$\overline{v_\phi^2} = \overline{v_\phi^2} + \sigma_\phi^2, \quad (58)$$

and rearranging (56), one finds

$$v_c^2 - \overline{v_\phi^2} = \sigma_\phi^2 - \sigma_R^2 - \frac{R}{v} \frac{\partial(v\sigma_R^2)}{\partial R} - \frac{R}{v} \frac{\partial(v\sigma_{Rz}^2)}{\partial z}. \quad (59)$$

Near the midplane, the tilt term is often neglected as a first approximation. Then

$$v_c^2 - \bar{v}_\phi^2 = \sigma_R^2 \left[-\frac{d \ln(v \sigma_R^2)}{d \ln R} - 1 + \frac{\sigma_\phi^2}{\sigma_R^2} \right]. \quad (60)$$

This difference is the asymmetric drift: random radial motions reduce the mean azimuthal streaming below the circular speed.

For a warm population with $v_a = v_c - \bar{v}_\phi \ll v_c$,

$$v_c^2 - \bar{v}_\phi^2 = (v_c - \bar{v}_\phi)(v_c + \bar{v}_\phi) \approx 2v_c v_a, \quad (61)$$

so

$$v_a \approx \frac{\sigma_R^2}{2v_c} \left[-\frac{d \ln(v \sigma_R^2)}{d \ln R} - 1 + \frac{\sigma_\phi^2}{\sigma_R^2} \right]. \quad (62)$$

For an exponential tracer density and dispersion profile,

$$v(R) = v_0 e^{-R/R_d}, \quad \sigma_R^2(R) = \sigma_{R0}^2 e^{-R/R_\sigma}, \quad (63)$$

the logarithmic derivative is

$$-\frac{d \ln(v \sigma_R^2)}{d \ln R} = \frac{R}{R_d} + \frac{R}{R_\sigma}. \quad (64)$$

If $R = 8 \text{ kpc}$, $R_d = 3 \text{ kpc}$, $R_\sigma = 7 \text{ kpc}$, $\sigma_R = 40 \text{ km s}^{-1}$, $v_c = 230 \text{ km s}^{-1}$, and $\sigma_\phi^2/\sigma_R^2 = 0.5$, then

$$\begin{aligned} v_a &= \frac{1600}{460} \left[\frac{8}{3} + \frac{8}{7} - 1 + 0.5 \right] \text{ km s}^{-1} \\ &= 11.5 \text{ km s}^{-1}. \end{aligned} \quad (65)$$

The result is of the same order as the observed lag of old thin-disk stars.

6.2. Vertical Jeans equation and local density

For a steady axisymmetric disk, the vertical Jeans equation is

$$\frac{1}{R} \frac{\partial(R v \sigma_{Rz}^2)}{\partial R} + \frac{\partial(v \sigma_z^2)}{\partial z} + v \frac{\partial \Phi}{\partial z} = 0. \quad (66)$$

If the tilt term is small near the solar cylinder,

$$\frac{\partial(v \sigma_z^2)}{\partial z} = -v K_z, \quad K_z = \frac{\partial \Phi}{\partial z}. \quad (67)$$

Thus the vertical force is

$$K_z(z) = -\frac{1}{v} \frac{\partial(v \sigma_z^2)}{\partial z}. \quad (68)$$

Combining with Poisson's equation in cylindrical coordinates gives

$$4\pi G \rho = \frac{1}{R} \frac{\partial}{\partial R} \left(R \frac{\partial \Phi}{\partial R} \right) + \frac{\partial^2 \Phi}{\partial z^2}. \quad (69)$$

Since

$$R \frac{\partial \Phi}{\partial R} = v_c^2(R), \quad (70)$$

the local density is

$$4\pi G \rho(R, z) = \frac{1}{R} \frac{\partial v_c^2}{\partial R} + \frac{\partial K_z}{\partial z}. \quad (71)$$

For a locally flat rotation curve, the radial term is small:

$$\frac{1}{R} \frac{\partial v_c^2}{\partial R} \approx 0. \quad (72)$$

The Oort-limit measurement is then essentially a measurement of $\partial K_z / \partial z$.

A standard local dark-matter density is

$$\rho_{\text{dm}, \odot} \approx 0.01 M_\odot \text{ pc}^{-3}, \quad (73)$$

equivalent to

$$\rho_{\text{dm}, \odot} \approx 0.38 \text{ GeV cm}^{-3}. \quad (74)$$

The conversion follows from

$$1 M_\odot \text{ pc}^{-3} = 37.9 \text{ GeV cm}^{-3}. \quad (75)$$

The local density is dynamically important because it calibrates direct-detection experiments and anchors Milky-Way mass models [12].

7. Rotation Curves and Elementary Mass Models

7.1. Point mass and luminous finite systems

For a point mass M ,

$$\Phi(r) = -\frac{GM}{r}, \quad (76)$$

and

$$v_c(r) = \sqrt{\frac{GM}{r}} \propto r^{-1/2}. \quad (77)$$

Any finite mass distribution behaves similarly at radii much larger than its extent:

$$M(r) \rightarrow M_{\text{tot}} \Rightarrow v_c(r) \rightarrow \sqrt{\frac{GM_{\text{tot}}}{r}}. \quad (78)$$

Therefore an extended flat rotation curve indicates that the enclosed mass continues to grow approximately linearly with radius.

7.2. Exponential disk

A thin stellar disk is often approximated by

$$\Sigma(R) = \Sigma_0 e^{-R/R_d}, \quad (79)$$

with total mass

$$M_d = 2\pi \Sigma_0 R_d^2. \quad (80)$$

The exact circular-speed expression involves modified Bessel functions; qualitatively, v_c rises near the centre, peaks at

$$R \approx 2.2 R_d, \quad (81)$$

and then declines approximately Keplerian when most disk mass is enclosed. Thus a disk alone cannot maintain an indefinitely flat curve unless its mass-to-light ratio rises strongly with radius.

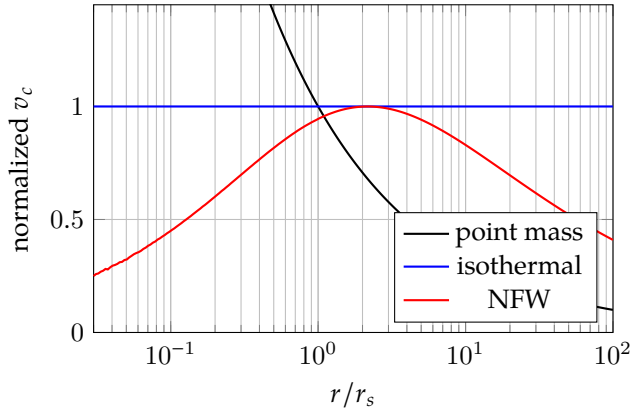


Figure 2. Representative rotation curves. A point mass declines as $r^{-1/2}$, an isothermal sphere is flat, and an NFW halo rises, peaks, and slowly declines.

7.3. Isothermal sphere

A singular isothermal sphere has density

$$\rho(r) = \frac{\sigma^2}{2\pi G r^2}. \quad (82)$$

The enclosed mass is

$$\begin{aligned} M(r) &= 4\pi \int_0^r \frac{\sigma^2}{2\pi G s^2} s^2 ds \\ &= \frac{2\sigma^2 r}{G}. \end{aligned} \quad (83)$$

The circular speed is therefore

$$v_c^2 = \frac{GM(r)}{r} = 2\sigma^2, \quad (84)$$

which is exactly flat. The singular isothermal sphere captures the observed flatness but has two defects: it is singular at $r = 0$ and has infinite total mass.

8. The NFW Halo

8.1. Density, mass, and slopes

Let

$$x = \frac{r}{r_s}. \quad (85)$$

The NFW density profile is

$$\rho(r) = \frac{\rho_s}{x(1+x)^2}. \quad (86)$$

Its logarithmic slope is

$$\begin{aligned} \frac{d \ln \rho}{d \ln r} &= \frac{d}{d \ln x} [-\ln x - 2 \ln(1+x)] \\ &= -1 - \frac{2x}{1+x}. \end{aligned} \quad (87)$$

Hence

$$\frac{d \ln \rho}{d \ln r} \rightarrow -1 \quad (x \ll 1), \quad (88)$$

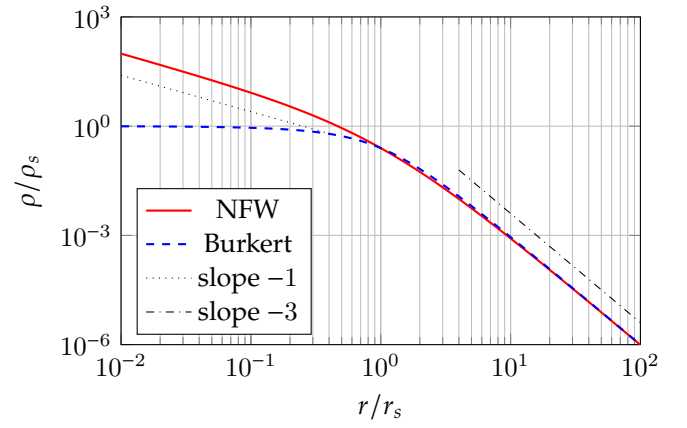


Figure 3. NFW and Burkert density profiles. The NFW halo is cusped, $\rho \propto r^{-1}$, while the Burkert halo has a finite central density. Both fall approximately as r^{-3} at large radius.

and

$$\frac{d \ln \rho}{d \ln r} \rightarrow -3 \quad (x \gg 1). \quad (89)$$

The mass follows from

$$\begin{aligned} M(r) &= 4\pi \rho_s r_s^3 \int_0^x \frac{y^2 dy}{y(1+y)^2} \\ &= 4\pi \rho_s r_s^3 \int_0^x \frac{y dy}{(1+y)^2}. \end{aligned} \quad (90)$$

Write

$$\frac{y}{(1+y)^2} = \frac{1}{1+y} - \frac{1}{(1+y)^2}. \quad (91)$$

Therefore

$$\begin{aligned} \int_0^x \frac{y dy}{(1+y)^2} &= \left[\ln(1+y) + \frac{1}{1+y} \right]_0^x \\ &= \ln(1+x) - \frac{x}{1+x}. \end{aligned} \quad (92)$$

Thus

$$M(r) = 4\pi \rho_s r_s^3 \left[\ln(1+x) - \frac{x}{1+x} \right]. \quad (93)$$

The mass diverges logarithmically as $r \rightarrow \infty$, so the profile must be truncated or interpreted within a virial radius.

8.2. Potential

For a spherical mass distribution with $\Phi(\infty) = 0$,

$$\Phi(r) = -G \int_r^\infty \frac{M(s)}{s^2} ds. \quad (94)$$

Substitute (93) with $u = s/r_s$:

$$\Phi(r) = -4\pi G \rho_s r_s^2 \int_x^\infty \frac{\ln(1+u) - u/(1+u)}{u^2} du. \quad (95)$$

Since

$$\frac{d}{du} \left[\frac{\ln(1+u)}{u} \right] = \frac{u/(1+u) - \ln(1+u)}{u^2}, \quad (96)$$

the integral is

$$\int_x^\infty \frac{\ln(1+u) - u/(1+u)}{u^2} du = \frac{\ln(1+x)}{x}. \quad (97)$$

Thus

$$\Phi(r) = -4\pi G \rho_s r_s^2 \frac{\ln(1+x)}{x}. \quad (98)$$

Equivalently,

$$\Phi(r) = -4\pi G \rho_s r_s^3 \frac{\ln(1+r/r_s)}{r}. \quad (99)$$

Although the density diverges as $1/r$, the potential is finite at the centre:

$$\Phi(0) = -4\pi G \rho_s r_s^2. \quad (100)$$

8.3. Circular speed and maximum velocity

Using (11) and (93),

$$v_c^2(r) = 4\pi G \rho_s r_s^2 \frac{\ln(1+x) - x/(1+x)}{x}. \quad (101)$$

Define

$$V_s^2 = 4\pi G \rho_s r_s^2. \quad (102)$$

Then

$$\frac{v_c^2}{V_s^2} = \frac{m(x)}{x}, \quad m(x) = \ln(1+x) - \frac{x}{1+x}. \quad (103)$$

The maximum occurs where

$$\frac{d}{dx} \left[\frac{m(x)}{x} \right] = 0. \quad (104)$$

Since

$$m'(x) = \frac{x}{(1+x)^2}, \quad (105)$$

the condition becomes

$$x m'(x) - m(x) = 0, \quad (106)$$

or

$$\frac{x^2}{(1+x)^2} - \ln(1+x) + \frac{x}{1+x} = 0. \quad (107)$$

Numerically,

$$x_{\max} = 2.16258. \quad (108)$$

At this radius,

$$V_{\max}^2 = V_s^2 \frac{m(2.16258)}{2.16258}. \quad (109)$$

The NFW curve declines only logarithmically slowly beyond the peak because $M(r) \propto \ln r$ at large radius.

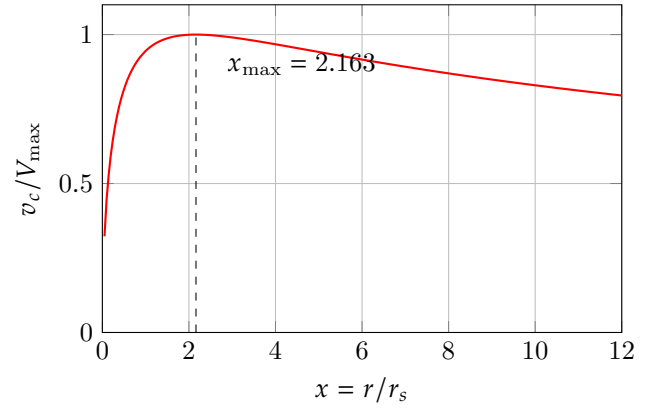


Figure 4. NFW circular-speed curve normalized by its maximum. The peak occurs at $r = 2.163r_s$.

8.4. Virial mass, concentration, and characteristic density

Define r_{200} as the radius enclosing mean density 200 times the critical density,

$$\bar{\rho}(< r_{200}) = 200\rho_{\text{crit}}. \quad (110)$$

The critical density is

$$\rho_{\text{crit}} = \frac{3H^2}{8\pi G}. \quad (111)$$

The virial mass is

$$M_{200} = \frac{4\pi}{3} 200\rho_{\text{crit}} r_{200}^3. \quad (112)$$

The concentration is

$$c = \frac{r_{200}}{r_s}. \quad (113)$$

Evaluating the NFW mass at r_{200} gives

$$M_{200} = 4\pi\rho_s r_s^3 \left[\ln(1+c) - \frac{c}{1+c} \right]. \quad (114)$$

Equating (112) and (114) yields

$$\rho_s = \frac{200}{3} \rho_{\text{crit}} \frac{c^3}{\ln(1+c) - c/(1+c)}. \quad (115)$$

It is common to define

$$\delta_c = \frac{\rho_s}{\rho_{\text{crit}}} = \frac{200}{3} \frac{c^3}{\ln(1+c) - c/(1+c)}. \quad (116)$$

The virial circular speed is

$$V_{200}^2 = \frac{GM_{200}}{r_{200}}, \quad (117)$$

and the NFW rotation curve may be written

$$\frac{v_c^2(r)}{V_{200}^2} = \frac{c}{x} \frac{\ln(1+x) - x/(1+x)}{\ln(1+c) - c/(1+c)}. \quad (118)$$

Table 1. Characteristic NFW quantities.

Quantity	Definition	Meaning
r_s	slope radius	$d \ln \rho / d \ln r = -2$
ρ_s	scale density	normalization
x	r/r_s	dimensionless radius
$m(x)$	$\ln(1+x) - x/(1+x)$	dimensionless mass
c	r_{200}/r_s	concentration
M_{200}	$(4\pi/3)200\rho_{\text{crit}}r_{200}^3$	virial mass
δ_c	$\rho_s/\rho_{\text{crit}}$	characteristic overdensity
V_{max}	$v_c(2.163r_s)$	peak halo speed

9. Worked Numerical Applications

9.1. A Milky-Way-like NFW halo

Take

$$M_{200} = 10^{12} M_{\odot}, \quad c = 10, \quad H_0 = 70 \text{ km s}^{-1} \text{ Mpc}^{-1}. \quad (119)$$

For this value of H_0 ,

$$\rho_{\text{crit}} \simeq 136 M_{\odot} \text{ kpc}^{-3}. \quad (120)$$

From (112),

$$\begin{aligned} r_{200} &= \left[\frac{3M_{200}}{4\pi 200\rho_{\text{crit}}} \right]^{1/3} \\ &= 206 \text{ kpc}. \end{aligned} \quad (121)$$

Then

$$r_s = \frac{r_{200}}{c} = 20.6 \text{ kpc}. \quad (122)$$

For $c = 10$,

$$m(c) = \ln(11) - \frac{10}{11} = 1.489. \quad (123)$$

Thus

$$\begin{aligned} \rho_s &= \frac{M_{200}}{4\pi r_s^3 m(c)} \\ &= 6.11 \times 10^6 M_{\odot} \text{ kpc}^{-3} \\ &= 6.11 \times 10^{-3} M_{\odot} \text{ pc}^{-3}. \end{aligned} \quad (124)$$

At the solar radius $R_0 = 8.2 \text{ kpc}$,

$$x_0 = \frac{8.2}{20.6} = 0.398. \quad (125)$$

The halo density is

$$\begin{aligned} \rho_h(R_0) &= \frac{6.11 \times 10^{-3}}{0.398(1.398)^2} M_{\odot} \text{ pc}^{-3} \\ &= 7.86 \times 10^{-3} M_{\odot} \text{ pc}^{-3}. \end{aligned} \quad (126)$$

This is close to the canonical local dark-matter density in (73); the difference between halo-only and full Galactic modelling is at the tens of percent level.

The virial speed is

$$V_{200} = \sqrt{\frac{GM_{200}}{r_{200}}} = 144 \text{ km s}^{-1}, \quad (127)$$

using $G = 4.302 \times 10^{-6} \text{ kpc}(\text{km s}^{-1})^2 M_{\odot}^{-1}$. The maximum speed is obtained from (118) at $x = 2.163$:

$$V_{\text{max}} = V_{200} \left[\frac{c}{2.163} \frac{m(2.163)}{m(10)} \right]^{1/2} = 174 \text{ km s}^{-1}. \quad (128)$$

9.2. A dwarf spheroidal mass-to-light ratio

For an ultra-faint dwarf with $R_e = 50 \text{ pc}$ and $\sigma_{\text{los}} = 4 \text{ km s}^{-1}$,

$$\begin{aligned} M(r_{1/2}) &= \frac{4R_e \sigma_{\text{los}}^2}{G} \\ &= \frac{4(50)(16)}{4.302 \times 10^{-3}} M_{\odot} \\ &= 7.44 \times 10^5 M_{\odot}. \end{aligned} \quad (129)$$

If $L = 3000 L_{\odot}$,

$$M/L \simeq 248 M_{\odot}/L_{\odot}. \quad (130)$$

This illustrates why dwarf spheroidals are among the most dark-matter-dominated known stellar systems.

9.3. Surface density from vertical kinematics

For a thin sheet approximation, the vertical force at height z is related to surface density by

$$K_z(z) \simeq 2\pi G \Sigma(|z|). \quad (131)$$

If kinematics give $\Sigma(1.1 \text{ kpc}) \simeq 70 M_{\odot} \text{ pc}^{-2}$ and visible matter contributes $55 M_{\odot} \text{ pc}^{-2}$, the dark contribution is

$$\Sigma_{\text{dm}} \simeq 15 M_{\odot} \text{ pc}^{-2}. \quad (132)$$

Approximating this as a roughly constant density over $|z| < 1.1 \text{ kpc}$ gives

$$\rho_{\text{dm}} \simeq \frac{\Sigma_{\text{dm}}}{2(1100 \text{ pc})} = 6.8 \times 10^{-3} M_{\odot} \text{ pc}^{-3}, \quad (133)$$

again of order $0.01 M_{\odot} \text{ pc}^{-3}$.

10. Cores, Cusps, and Alternative Halo Profiles

The Burkert profile is a commonly used cored alternative:

$$\rho_B(r) = \frac{\rho_0}{(1+r/r_0)[1+(r/r_0)^2]}. \quad (134)$$

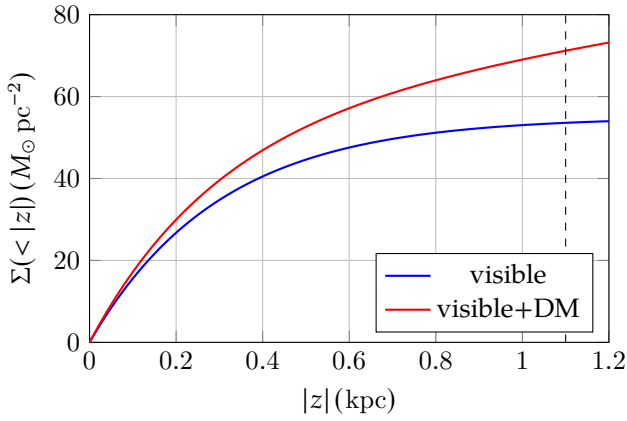


Figure 5. Illustrative vertical surface-density budget near the solar radius. A nearly constant dark-matter density adds a component growing approximately linearly with height.

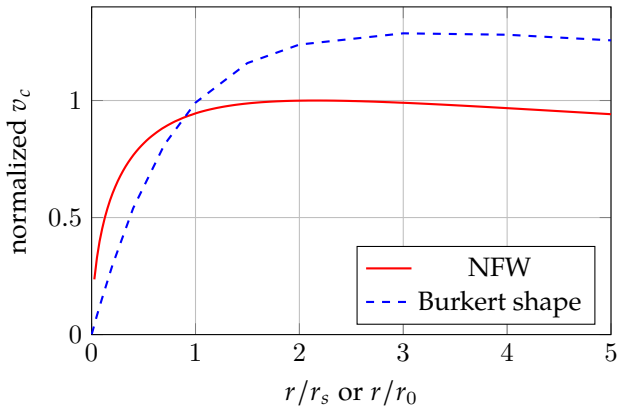


Figure 6. Inner circular-speed shapes for cusped and cored profiles. The Burkert curve rises more nearly linearly at small radius, while the NFW curve rises as $r^{1/2}$.

At small radii,

$$\rho_B(r) \rightarrow \rho_0, \quad (135)$$

whereas the NFW profile has $\rho \propto r^{-1}$. For the Burkert profile the inner enclosed mass scales as

$$M_B(r) \propto r^3 \quad (r \ll r_0), \quad (136)$$

and therefore

$$v_c(r) \propto r. \quad (137)$$

For an NFW cusp,

$$M_{\text{NFW}}(r) \propto r^2 \quad (r \ll r_s), \quad (138)$$

so

$$v_c(r) \propto r^{1/2}. \quad (139)$$

The difference between solid-body rise and square-root rise is the kinematic signature of a core versus a cusp, though observational beam smearing, non-circular motions, baryonic feedback, and inclination uncertainties can obscure it.

11. Discussion

The Jeans equations are powerful because they require only low-order velocity moments, but this economy is

also their principal weakness. A Jeans solution need not correspond to a non-negative distribution function. Even when it does, projected observables may not uniquely determine $M(r)$, $\beta(r)$, and the tracer density. The spherical mass-anisotropy degeneracy is the classic example: radial anisotropy can mimic a change in mass profile. Proper motions, higher velocity moments, multiple tracer populations, and distribution-function modelling help break this degeneracy.

Rotation curves are more direct but not assumption-free. Gas rotation is affected by pressure gradients, turbulence, warps, bars, spiral streaming, and the finite thickness of the disk. Stellar rotation requires asymmetric-drift corrections. Decomposition into bulge, disk, gas, and halo is limited by the stellar mass-to-light ratio. Nevertheless, the empirical flatness found by Rubin and Bosma remains one of the clearest dynamical arguments for dark matter: beyond the luminous disk, $v_c \simeq \text{constant}$ implies $M(r) \propto r$, far exceeding the visible mass.

The NFW profile is a cornerstone of the cold-dark-matter paradigm. Its two parameters may be replaced by M_{200} and c , connecting galaxy dynamics to cosmology. In hierarchical structure formation, concentration reflects the characteristic density of the Universe when the halo assembled: low-mass halos typically form earlier and are more concentrated. Precision cosmology fixes the background parameters entering ρ_{crit} [13], while simulations calibrate the mass-concentration relation.

Yet the NFW form is not universally sufficient. The cusp-core problem asks why some dark-matter-dominated galaxies appear to have central cores. Possible explanations include baryonic feedback that transfers energy to dark matter through repeated gas inflow and outflow, self-interacting dark matter, fuzzy or warm dark matter, and systematic observational effects. Conversely, some systems are consistent with cusps. The problem is therefore not merely whether NFW is wrong, but under what conditions collisionless cold dark matter plus baryonic physics predicts departures from it.

The local dark-matter density illustrates another modern frontier. Gaia astrometry has transformed vertical Jeans analyses by providing phase-space information for millions of stars. However, disequilibria from spiral arms, the Galactic bar, satellite perturbations, and phase-space spirals violate the simplest steady-state assumptions. The local value $\rho_{\text{dm},\odot} \sim 0.01 M_\odot \text{pc}^{-3}$ is robust at order unity, but percent-level precision requires non-equilibrium modelling.

12. Conclusion

Starting from phase-space number conservation, we derived the collisionless Boltzmann equation and its velocity moments. The zeroth moment gives tracer continuity, while the first moment gives the Jeans equations, in which gravity is balanced by gradients of velocity

dispersion and streaming motion. In spherical systems the Jeans equation relates the enclosed mass to tracer density, radial dispersion, and anisotropy; in disks the radial equation yields asymmetric drift and the vertical equation gives the local density through the vertical force.

Rotation curves translate these dynamical principles into one of the strongest arguments for dark matter. A point mass gives $v_c \propto r^{-1/2}$, an exponential disk peaks and declines, whereas observed spiral galaxies commonly show nearly flat curves. An isothermal sphere explains flatness but is singular and has infinite mass. The NFW profile, motivated by cosmological simulations, provides a physically connected halo model with density slopes -1 and -3 , enclosed mass $M(r) = 4\pi\rho_s r_s^3 [\ln(1+x) - x/(1+x)]$, potential $\Phi = -4\pi G\rho_s r_s^3 \ln(1+r/r_s)/r$, and a circular-speed maximum at $r = 2.163r_s$.

Jeans methods and NFW halos remain indispensable, but they are not final answers. Their assumptions must be tested against non-equilibrium structure, baryonic effects, and high-quality data. The continuing dialogue between theory, simulations, and observations is precisely what makes galactic dynamics a central discipline in modern astrophysics.

References

- [1] J. H. Jeans, "On the theory of star-streaming and the structure of the universe," *Monthly Notices of the Royal Astronomical Society*, vol. 76, pp. 70–84, 1915.
- [2] A. S. Eddington, "The distribution of stars in globular clusters," *Monthly Notices of the Royal Astronomical Society*, vol. 76, pp. 572–585, 1916.
- [3] S. Chandrasekhar, *Principles of Stellar Dynamics*. University of Chicago Press, 1942.
- [4] J. Binney and S. Tremaine, *Galactic Dynamics*, 2nd ed. Princeton University Press, 2008.
- [5] B. W. Carroll and D. A. Ostlie, *An Introduction to Modern Astrophysics*, 2nd ed. Cambridge University Press, 2017.
- [6] V. C. Rubin, W. K. Ford Jr., and N. Thonnard, "Rotational properties of 21 Sc galaxies with a large range of luminosities and radii," *Astrophysical Journal*, vol. 238, pp. 471–487, 1980.
- [7] A. Bosma, "21-cm line studies of spiral galaxies. II. The distribution and kinematics of neutral hydrogen in spiral galaxies of various morphological types," *Astronomical Journal*, vol. 86, pp. 1825–1846, 1981.
- [8] J. F. Navarro, C. S. Frenk, and S. D. M. White, "The structure of cold dark matter halos," *Astrophysical Journal*, vol. 462, pp. 563–575, 1996.
- [9] J. F. Navarro, C. S. Frenk, and S. D. M. White, "A universal density profile from hierarchical clustering," *Astrophysical Journal*, vol. 490, pp. 493–508, 1997.
- [10] A. Burkert, "The structure of dark matter halos in dwarf galaxies," *Astrophysical Journal Letters*, vol. 447, pp. L25–L28, 1995.
- [11] J. Wolf, G. D. Martinez, J. S. Bullock, M. Kaplinghat, M. Geha, R. R. Muñoz, J. D. Simon, and F. F. Avedo, "Accurate masses for dispersion-supported galaxies," *Monthly Notices of the Royal Astronomical Society*, vol. 406, pp. 1220–1237, 2010.
- [12] J. Bovy and S. Tremaine, "On the local dark matter density," *Astrophysical Journal*, vol. 756, article 89, 2012.
- [13] Planck Collaboration, "Planck 2018 results. VI. Cosmological parameters," *Astronomy & Astrophysics*, vol. 641, article A6, 2020.
- [14] W. J. G. de Blok, "The core-cusp problem," *Advances in Astronomy*, vol. 2010, article 789293, 2010.
- [15] J. I. Read, G. Iorio, O. Agertz, and F. Fraternali, "Understanding the shape and diversity of dwarf galaxy rotation curves in Λ CDM," *Monthly Notices of the Royal Astronomical Society*, vol. 462, pp. 3628–3645, 2016.