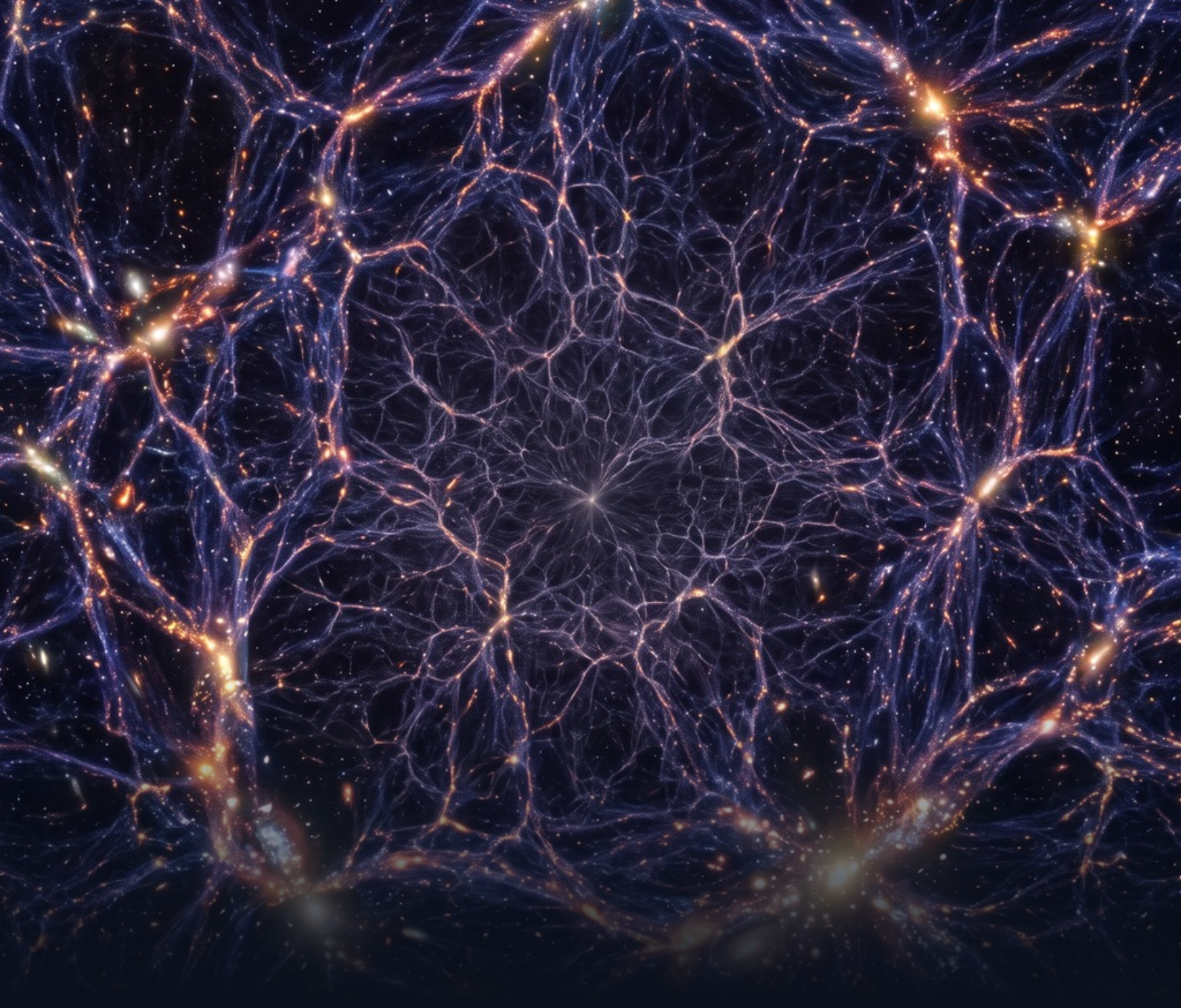




COSMOLOGY

Derivation of the Friedmann Equations from the FLRW Metric and the Accelerating Expansion

Academic Research Paper

Theoretical Astrophysics — Step-by-Step Analytic Derivations, Vector Diagrams and Numerical Tables

Derivation of the Friedmann Equations from the FLRW Metric and the Accelerating Expansion

Cosmology

September 2026

Abstract. The Friedmann equations are the dynamical core of modern homogeneous cosmology. They connect the geometry of an expanding spacetime to the density, pressure, curvature, and vacuum energy content of the Universe, and thereby provide the theoretical basis for the hot Big Bang, the cosmic age scale, the interpretation of redshift-distance observations, and the discovery of accelerated expansion. This paper gives a step-by-step derivation of the Friedmann equations from the Friedmann–Lemaître–Robertson–Walker metric, using explicit Christoffel symbols, Ricci components, and the perfect-fluid stress-energy tensor in the comoving frame. The fluid equation is derived from covariant energy conservation and shown to be equivalent to the time derivative of the first Friedmann equation together with the acceleration equation. We then discuss equations of state, density parameters, analytic scale-factor solutions, the age integral, the deceleration parameter, luminosity distances, and the Type Ia supernova Hubble diagram. Additional worked examples evaluate the cosmic age, critical density, transition redshift, and luminosity distance of a supernova at $z = 1$. We finally discuss extensions, observational tests, assumptions, and open issues including the Hubble tension.

Keywords: cosmology; Friedmann equations; FLRW metric; dark energy; cosmic acceleration; Type Ia supernovae

1. Introduction

Modern physical cosmology rests on the remarkable empirical fact that the Universe is close to homogeneous and isotropic when averaged on sufficiently large scales. Galaxy surveys show strong local clustering, filaments, walls, and voids, yet the statistical distribution approaches uniformity on scales of order 100 Mpc and larger. The cosmic microwave background radiation is still more striking: after removal of the dipole and foregrounds, its temperature anisotropies are of order 10^{-5} . These observations motivate the cosmological principle, according to which no spatial position and no spatial direction are privileged on large scales.

The mathematical consequence of this principle is severe. The spacetime metric must have spatial hypersurfaces of constant curvature, and its time evolution is described by a single function, the scale factor $a(t)$. The resulting Friedmann–Lemaître–Robertson–Walker (FLRW) metric is therefore the most general metric compatible with exact large-scale homogeneity and isotropy. Once this metric is inserted into Einstein’s field equations, the full tensor theory reduces to two ordinary differential equations for $a(t)$: the Friedmann equation and the acceleration equation. These equations were first derived in the 1920s by Friedmann and Lemaître and later placed in their modern geometrical form by Robertson and Walker [2, 3, 4, 5].

The Friedmann equations explain three fundamental cosmological observations. First, the recession of galaxies is not ordinary motion through a pre-existing Euclidean space, but the increase of proper distance between comoving worldlines as $a(t)$ grows. Second, the hot early Universe follows because radiation and matter densities increase rapidly as a decreases. Third, the late-time accelerated expansion discovered using Type

Ia supernovae [14, 15] implies that the cosmic energy budget contains a component with sufficiently negative pressure, most simply a cosmological constant Λ .

The aim of this paper is pedagogical but rigorous. We derive the equations directly from the metric, retaining factors of c and the spatial curvature parameter k . We then connect the formalism to observations by introducing density parameters, analytic limiting solutions, the luminosity distance, the age of the Universe, and numerical values from Planck 2018 [16]. The level and notation are chosen to be compatible with standard graduate treatments and advanced astrophysics texts such as Carroll and Ostlie [1].

2. Physical and Mathematical Framework

2.1. The cosmological principle

Definition 1 (Cosmological principle). On sufficiently large spatial scales the Universe is statistically homogeneous and isotropic. In the idealized exact limit, every spatial point is equivalent to every other point and every spatial direction is equivalent to every other direction.

Homogeneity eliminates preferred positions; isotropy about every point eliminates preferred directions. Together they imply that spatial slices of constant cosmic time have constant curvature. Such three-geometries are classified by a curvature index

$$k = \begin{cases} +1, & \text{closed positive-curvature space,} \\ 0, & \text{spatially flat Euclidean space,} \\ -1, & \text{open negative-curvature space.} \end{cases} \quad (1)$$

The dimensional curvature radius is absorbed into the

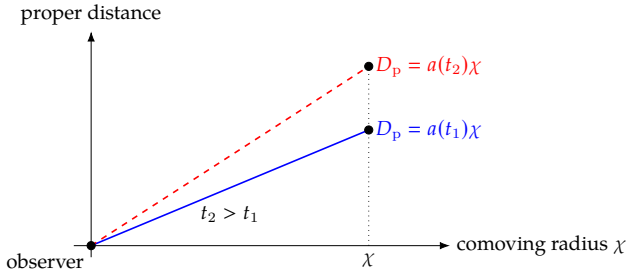


Figure 1. Comoving coordinates label fundamental observers. Their coordinate separation χ is fixed, while the proper distance grows in proportion to the scale factor $a(t)$.

scale factor $a(t)$ if $k = \pm 1$. If $k = 0$, the normalization of a is conventional.

2.2. FLRW line element

The FLRW line element in spherical comoving coordinates (t, r, θ, ϕ) is

$$ds^2 = c^2 dt^2 - a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right]. \quad (2)$$

The coordinate r is comoving: freely falling galaxies at rest in the smooth cosmic fluid have fixed r, θ, ϕ . The physical, or proper, radial distance at cosmic time t is

$$D_p(t) = a(t)\chi, \quad (3)$$

where the radial comoving coordinate χ is defined by

$$d\chi = \frac{dr}{\sqrt{1 - kr^2}}. \quad (4)$$

Thus

$$\chi(r) = \begin{cases} \sin^{-1} r, & k = +1, \\ r, & k = 0, \\ \sinh^{-1} r, & k = -1. \end{cases} \quad (5)$$

Differentiating (3) for fixed χ gives Hubble's law in its exact homogeneous form:

$$v_{\text{rec}} = \dot{D}_p = \dot{a}\chi = \frac{\dot{a}}{a} D_p. \quad (6)$$

The Hubble parameter is therefore

$$H(t) \equiv \frac{\dot{a}(t)}{a(t)}. \quad (7)$$

The redshift of a photon emitted when the scale factor was a_e and observed when it is a_0 is

$$1 + z = \frac{a_0}{a_e}. \quad (8)$$

It is conventional to set $a_0 = 1$ today.

The redshift relation can also be obtained directly from successive wave crests. For radial null rays, the emission and observation times obey

$$\int_{t_e}^{t_0} \frac{c dt}{a(t)} = \int_{t_e + \delta t_e}^{t_0 + \delta t_0} \frac{c dt}{a(t)}. \quad (9)$$

Subtracting the two nearly equal integrals gives

$$\frac{\delta t_0}{a_0} = \frac{\delta t_e}{a_e}, \quad (10)$$

and therefore

$$1 + z = \frac{v_e}{v_0} = \frac{\delta t_0}{\delta t_e} = \frac{a_0}{a_e}. \quad (11)$$

This derivation emphasizes that cosmological redshift is a metric effect along null geodesics, not a sequence of local Doppler shifts in a static background.

3. Geometry of the FLRW Metric

3.1. Metric components

It is useful to write the spatial part of (2) as

$$d\ell^2 = \gamma_{ij} dx^i dx^j = \frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2, \quad (12)$$

so that the four-metric has components

$$g_{00} = c^2, \quad g_{0i} = 0, \quad g_{ij} = -a^2(t) \gamma_{ij}. \quad (13)$$

The inverse metric is

$$g^{00} = \frac{1}{c^2}, \quad g^{0i} = 0, \quad g^{ij} = -\frac{1}{a^2(t)} \gamma^{ij}. \quad (14)$$

The determinant separates into temporal and spatial factors:

$$\sqrt{-g} = c a^3 \sqrt{\gamma} = \frac{c a^3 r^2 \sin \theta}{\sqrt{1 - kr^2}}. \quad (15)$$

This factor is useful when checking covariant conservation and when constructing volume elements. The proper volume element on a constant-time hypersurface is

$$dV_p = a^3 \frac{r^2 \sin \theta}{\sqrt{1 - kr^2}} dr d\theta d\phi. \quad (16)$$

3.2. Christoffel symbols

The Levi-Civita connection is

$$\Gamma_{\mu\nu}^\alpha = \frac{1}{2} g^{\alpha\beta} (\partial_\mu g_{\nu\beta} + \partial_\nu g_{\mu\beta} - \partial_\beta g_{\mu\nu}). \quad (17)$$

Because only $a(t)$ depends on time,

$$\partial_0 g_{ij} = -2a\dot{a} \gamma_{ij}, \quad (18)$$

where the overdot denotes d/dt . The mixed time-space symbols are

$$\Gamma_{ij}^0 = -\frac{1}{2} g^{00} \partial_0 g_{ij} = \frac{a\dot{a}}{c^2} \gamma_{ij}, \quad (19)$$

$$\Gamma_{0j}^i = \frac{1}{2} g^{ik} \partial_0 g_{jk} = \frac{\dot{a}}{a} \delta_j^i = H \delta_j^i. \quad (20)$$

The purely temporal symbols vanish:

$$\Gamma_{00}^0 = 0, \quad \Gamma_{00}^i = 0, \quad \Gamma_{0i}^0 = 0. \quad (21)$$

The purely spatial symbols equal those of the constant-curvature metric γ_{ij} :

$$\Gamma^i_{jk} = {}^{(3)}\Gamma^i_{jk}(\gamma). \quad (22)$$

Explicitly,

$${}^{(3)}\Gamma^r_{rr} = \frac{kr}{1-kr^2}, \quad {}^{(3)}\Gamma^r_{\theta\theta} = -r(1-kr^2), \quad (23)$$

$${}^{(3)}\Gamma^r_{\phi\phi} = -r(1-kr^2)\sin^2\theta, \quad {}^{(3)}\Gamma^\theta_{r\theta} = \frac{1}{r}, \quad (24)$$

$${}^{(3)}\Gamma^\phi_{r\phi} = \frac{1}{r}, \quad {}^{(3)}\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta, \quad (25)$$

$${}^{(3)}\Gamma^\phi_{\theta\phi} = \cot\theta. \quad (26)$$

3.3. Ricci tensor and scalar

With the curvature convention

$$R^\rho_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma}, \quad (27)$$

the Ricci tensor is

$$R_{\mu\nu} = R^\alpha_{\mu\alpha\nu}. \quad (28)$$

For the time-time component,

$$R_{00} = -\partial_0 \Gamma^\alpha_{0\alpha} - \Gamma^\alpha_{0\beta} \Gamma^\beta_{0\alpha} \quad (29)$$

$$= -\frac{d}{dt}(3H) - 3H^2 = -3(\dot{H} + H^2) \quad (30)$$

$$= -3\frac{\ddot{a}}{a}. \quad (31)$$

For the spatial components one obtains

$$R_{ij} = \left(\frac{a\ddot{a}}{c^2} + \frac{2\dot{a}^2}{c^2} + 2k \right) \gamma_{ij}. \quad (32)$$

Equivalently, since $g_{ij} = -a^2\gamma_{ij}$,

$$R_{ij} = - \left[\frac{\ddot{a}}{ac^2} + \frac{2H^2}{c^2} + \frac{2k}{a^2} \right] g_{ij}. \quad (33)$$

The mixed time-space components vanish:

$$R_{0i} = 0. \quad (34)$$

Contracting gives the Ricci scalar

$$R = g^{00}R_{00} + g^{ij}R_{ij} \quad (35)$$

$$= -6 \left[\frac{\ddot{a}}{ac^2} + \frac{H^2}{c^2} + \frac{k}{a^2} \right]. \quad (36)$$

The purely spatial metric γ_{ij} has intrinsic Ricci tensor and scalar

$${}^{(3)}R_{ij}(\gamma) = 2k \gamma_{ij}, \quad (37)$$

$${}^{(3)}R(\gamma) = 6k. \quad (38)$$

For the physical spatial metric $a^2\gamma_{ij}$, lengths are multiplied by a and curvature scales as inverse length squared, so

$${}^{(3)}R_{\text{phys}} = \frac{6k}{a^2}. \quad (39)$$

Equation (39) is the geometrical origin of the kc^2/a^2 term in the Friedmann constraint.

The Einstein tensor is

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}. \quad (40)$$

The nonzero independent FLRW components are

$$G_{00} = 3 \left(H^2 + \frac{kc^2}{a^2} \right), \quad (41)$$

$$G_{ij} = - \left(2\frac{\ddot{a}}{a} + H^2 + \frac{kc^2}{a^2} \right) g_{ij}. \quad (42)$$

Equation (41) is the geometrical origin of the first Friedmann equation, while (42) is the origin of the acceleration equation.

4. Matter Source: Perfect Fluid

On large scales all matter and radiation are approximated by a perfect fluid. In the local rest frame its stress-energy tensor is diagonal:

$$T^{\hat{\mu}}_{\hat{\nu}} = \text{diag}(\rho c^2, -p, -p, -p), \quad (43)$$

where ρ is mass-energy density divided by c^2 and p is isotropic pressure. Covariantly,

$$T_{\mu\nu} = \left(\rho + \frac{p}{c^2} \right) u_\mu u_\nu - p g_{\mu\nu}, \quad (44)$$

with

$$u^\mu u_\mu = c^2. \quad (45)$$

For a comoving observer,

$$u^\mu = (1, 0, 0, 0), \quad u_\mu = (c^2, 0, 0, 0). \quad (46)$$

Hence

$$T_{00} = \rho c^4, \quad (47)$$

$$T_{ij} = p a^2 \gamma_{ij} = -p g_{ij}. \quad (48)$$

The pressure contributes to gravity because it appears in the spatial components of $T_{\mu\nu}$ and therefore affects spacetime curvature.

The trace of the perfect-fluid stress-energy tensor is

$$T = g^{\mu\nu} T_{\mu\nu} = \rho c^2 - 3p. \quad (49)$$

Thus radiation, for which $p = \rho c^2/3$, has a traceless stress-energy tensor:

$$T_{\text{rad}} = 0. \quad (50)$$

This fact is closely related to the conformal invariance of freely propagating electromagnetic radiation in four spacetime dimensions.

5. Derivation of the Friedmann Equations

We adopt the sign convention

$$G_{\mu\nu} - \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}. \quad (51)$$

With the curvature convention (27), this yields the standard cosmological signs used below.

5.1. First Friedmann equation

The 00 component of (51) is

$$G_{00} - \Lambda g_{00} = \frac{8\pi G}{c^4} T_{00}. \quad (52)$$

Using (41), (47), and $g_{00} = c^2$ gives

$$3 \left(H^2 + \frac{kc^2}{a^2} \right) - \Lambda c^2 = 8\pi G \rho. \quad (53)$$

Solving for H^2 gives the first Friedmann equation:

$$H^2 = \frac{8\pi G}{3} \rho - \frac{kc^2}{a^2} + \frac{\Lambda c^2}{3}. \quad (54)$$

Remark 1. The terms in (54) have transparent meanings: matter and radiation density increase H^2 , positive spatial curvature decreases H^2 , negative curvature increases it, and a positive cosmological constant behaves as a constant energy density that accelerates expansion.

5.2. Second Friedmann equation

The spatial part of (51) is

$$G_{ij} - \Lambda g_{ij} = \frac{8\pi G}{c^4} T_{ij}. \quad (55)$$

Substitution of (42) and (48) gives the independent spatial equation

$$2 \frac{\ddot{a}}{a} + H^2 + \frac{kc^2}{a^2} - \Lambda c^2 = -\frac{8\pi G}{c^2} p. \quad (56)$$

Eliminating $H^2 + kc^2/a^2$ using (54) gives

$$2 \frac{\ddot{a}}{a} + \frac{8\pi G}{3} \rho + \frac{\Lambda c^2}{3} - \Lambda c^2 = -\frac{8\pi G}{c^2} p, \quad (57)$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \left(\rho + \frac{3p}{c^2} \right) + \frac{\Lambda c^2}{3}. \quad (58)$$

Equation (58) is the acceleration equation.

Remark 2. The active gravitational mass density is $\rho + 3p/c^2$. Radiation gravitates more strongly than dust because $p = \rho c^2/3$. Vacuum energy has $p = -\rho c^2$, making $\rho + 3p/c^2 = -2\rho$, and therefore drives acceleration.

5.3. Fluid equation from covariant conservation

The contracted Bianchi identity requires

$$\nabla_\mu G^{\mu\nu} = 0. \quad (59)$$

Since $\nabla_\mu g^{\mu\nu} = 0$, (51) implies local stress-energy conservation:

$$\nabla_\mu T^{\mu\nu} = 0. \quad (60)$$

For $\nu = 0$ in the comoving FLRW frame,

$$\dot{\rho} + 3H \left(\rho + \frac{p}{c^2} \right) = 0. \quad (61)$$

The same equation follows thermodynamically. The energy in a comoving volume $V \propto a^3$ is

$$E = \rho c^2 a^3. \quad (62)$$

Adiabatic expansion obeys

$$dE = -p d(a^3). \quad (63)$$

Taking a time derivative,

$$c^2 \frac{d}{dt}(\rho a^3) = -p \frac{d}{dt}(a^3). \quad (64)$$

Expanding both sides gives

$$c^2 a^3 \dot{\rho} + 3c^2 a^2 \dot{a} \rho = -3p a^2 \dot{a}, \quad (65)$$

and division by $c^2 a^3$ gives (61).

5.4. Equivalence of the equations

Only two of (54), (58), and (61) are independent. Differentiate (54):

$$2H\dot{H} = \frac{8\pi G}{3} \dot{\rho} + \frac{2kc^2 \dot{a}}{a^3}. \quad (66)$$

Using (61),

$$2H\dot{H} = -8\pi GH \left(\rho + \frac{p}{c^2} \right) + \frac{2kc^2 \dot{H}}{a^2}. \quad (67)$$

For $H \neq 0$,

$$\dot{H} = -4\pi G \left(\rho + \frac{p}{c^2} \right) + \frac{kc^2}{a^2}. \quad (68)$$

Since

$$\frac{\ddot{a}}{a} = \dot{H} + H^2, \quad (69)$$

substitution of (54) and (68) yields

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \left(\rho + \frac{3p}{c^2} \right) + \frac{\Lambda c^2}{3}, \quad (70)$$

which is precisely (58).

6. Equations of State and Density Scaling

A barotropic cosmic component is often described by

$$p = w\rho c^2, \quad (71)$$

where w is constant. The fluid equation becomes

$$\dot{\rho} + 3H\rho(1+w) = 0. \quad (72)$$

Using $H = \dot{a}/a$,

$$\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a}. \quad (73)$$

Integrating,

$$\ln\left(\frac{\rho}{\rho_0}\right) = -3(1+w)\ln\left(\frac{a}{a_0}\right), \quad (74)$$

so

$$\rho(a) = \rho_0 \left(\frac{a}{a_0}\right)^{-3(1+w)}. \quad (75)$$

Important cases are

$$w = 0 : \quad \rho_m \propto a^{-3}, \quad (76)$$

$$w = \frac{1}{3} : \quad \rho_r \propto a^{-4}, \quad (77)$$

$$w = -1 : \quad \rho_\Lambda = \text{constant}. \quad (78)$$

Matter dilutes by volume expansion. Radiation dilutes by volume and photon redshift. Vacuum energy remains constant because its negative pressure performs work that exactly compensates the increasing volume.

The cosmological constant may be written as an effective fluid:

$$\rho_\Lambda = \frac{\Lambda c^2}{8\pi G}, \quad (79)$$

$$p_\Lambda = -\rho_\Lambda c^2, \quad (80)$$

$$w_\Lambda = -1. \quad (81)$$

If w is not constant, the same separation gives the more general expression

$$\rho(a) = \rho_0 \exp\left[-3 \int_1^a (1+w(a')) \frac{da'}{a'}\right]. \quad (82)$$

A common phenomenological form is

$$w(a) = w_0 + w_a(1-a), \quad (83)$$

for which (82) becomes

$$\rho_{\text{de}}(a) = \rho_{\text{de},0} a^{-3(1+w_0+w_a)} \exp[-3w_a(1-a)]. \quad (84)$$

The Λ CDM limit is recovered by setting $w_0 = -1$ and $w_a = 0$.

7. Density Parameters and the Sum Rule

The critical density is defined by

$$\rho_c(t) = \frac{3H^2(t)}{8\pi G}. \quad (85)$$

For each component,

$$\Omega_i(t) = \frac{\rho_i(t)}{\rho_c(t)}. \quad (86)$$

The cosmological constant density parameter is

$$\Omega_\Lambda(t) = \frac{\Lambda c^2}{3H^2(t)}. \quad (87)$$

The curvature density parameter is defined by

$$\Omega_k(t) = -\frac{kc^2}{a^2 H^2}. \quad (88)$$

Dividing (54) by H^2 gives

$$1 = \Omega_m + \Omega_r + \Omega_\Lambda + \Omega_k. \quad (89)$$

At $a_0 = 1$, the expansion function is

$$E(a) \equiv \frac{H(a)}{H_0}. \quad (90)$$

For matter, radiation, curvature, and Λ ,

$$E^2(a) = \Omega_{r0}a^{-4} + \Omega_{m0}a^{-3} + \Omega_{k0}a^{-2} + \Omega_{\Lambda0}. \quad (91)$$

Equivalently in redshift,

$$E^2(z) = \Omega_{r0}(1+z)^4 + \Omega_{m0}(1+z)^3 + \Omega_{k0}(1+z)^2 + \Omega_{\Lambda0}. \quad (92)$$

The time-dependent fractional densities are

$$\Omega_m(a) = \frac{\Omega_{m0}a^{-3}}{E^2(a)}, \quad (93)$$

$$\Omega_r(a) = \frac{\Omega_{r0}a^{-4}}{E^2(a)}, \quad (94)$$

$$\Omega_\Lambda(a) = \frac{\Omega_{\Lambda0}}{E^2(a)}, \quad (95)$$

$$\Omega_k(a) = \frac{\Omega_{k0}a^{-2}}{E^2(a)}. \quad (96)$$

Radiation–matter equality occurs when $\rho_r(a) = \rho_m(a)$, namely

$$a_{\text{eq}} = \frac{\Omega_{r0}}{\Omega_{m0}}, \quad z_{\text{eq}} = \frac{\Omega_{m0}}{\Omega_{r0}} - 1. \quad (97)$$

Matter–vacuum equality occurs when $\rho_m(a) = \rho_\Lambda$, giving

$$a_{m\Lambda} = \left(\frac{\Omega_{m0}}{\Omega_{\Lambda0}}\right)^{1/3}, \quad z_{m\Lambda} = \left(\frac{\Omega_{\Lambda0}}{\Omega_{m0}}\right)^{1/3} - 1. \quad (98)$$

This equality is not the same as the onset of acceleration; acceleration requires the stronger condition $\rho_m < 2\rho_\Lambda$ in a flat matter-plus- Λ universe.

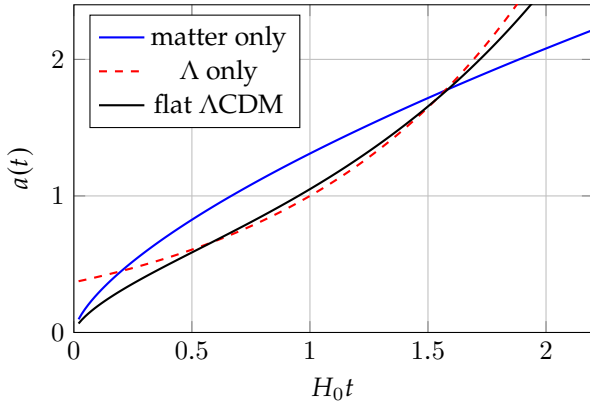


Figure 2. Scale-factor histories. Matter domination gives $a \propto t^{2/3}$, pure vacuum energy gives exponential expansion, and flat Λ CDM interpolates from matter-like early behavior to late acceleration.

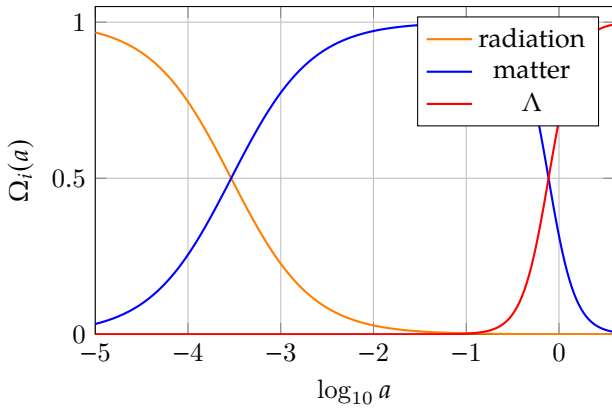


Figure 3. Evolution of fractional densities for a nearly flat Planck-like model. Radiation dominates at very small a , matter dominates at intermediate a , and vacuum energy dominates at late times.

8. Analytic Scale-Factor Solutions

8.1. Matter domination

For a flat, matter-only universe,

$$H^2 = H_0^2 a^{-3}. \quad (99)$$

Since $H = \dot{a}/a$,

$$\dot{a} = H_0 a^{-1/2}. \quad (100)$$

Separating variables,

$$a^{1/2} da = H_0 dt. \quad (101)$$

Integration with $a = 0$ at $t = 0$ gives

$$\frac{2}{3} a^{3/2} = H_0 t. \quad (102)$$

Therefore

$$a(t) = \left(\frac{3H_0 t}{2} \right)^{2/3}. \quad (103)$$

Thus matter-dominated expansion decelerates:

$$\ddot{a} < 0. \quad (104)$$

8.2. Radiation domination

For a flat, radiation-only universe,

$$H^2 = H_0^2 a^{-4}. \quad (105)$$

Thus

$$\dot{a} = H_0 a^{-1}. \quad (106)$$

Separating and integrating,

$$a da = H_0 dt, \quad (107)$$

so

$$\frac{1}{2} a^2 = H_0 t. \quad (108)$$

Hence

$$a(t) = (2H_0 t)^{1/2}. \quad (109)$$

Radiation decelerates more strongly than matter because pressure contributes positively to gravity.

8.3. Vacuum domination

For pure positive Λ ,

$$H^2 = \frac{\Lambda c^2}{3} \equiv H_\Lambda^2. \quad (110)$$

Hence

$$\frac{\dot{a}}{a} = H_\Lambda. \quad (111)$$

Integration gives

$$a(t) = a_i \exp [H_\Lambda (t - t_i)]. \quad (112)$$

The acceleration is

$$\frac{\ddot{a}}{a} = H_\Lambda^2 > 0. \quad (113)$$

8.4. Flat matter plus cosmological constant

For a spatially flat universe with pressureless matter and Λ ,

$$H^2 = H_0^2 (\Omega_m a^{-3} + \Omega_\Lambda), \quad (114)$$

with

$$\Omega_m + \Omega_\Lambda = 1. \quad (115)$$

The differential equation is

$$\frac{\dot{a}}{a} = H_0 (\Omega_m a^{-3} + \Omega_\Lambda)^{1/2}. \quad (116)$$

Let

$$y = a^{3/2}. \quad (117)$$

Then

$$\dot{y} = \frac{3}{2} a^{1/2} \dot{a}. \quad (118)$$

Using (116),

$$\dot{y} = \frac{3}{2} H_0 (\Omega_m + \Omega_\Lambda y^2)^{1/2}. \quad (119)$$

Separate:

$$\frac{dy}{\sqrt{\Omega_m + \Omega_\Lambda y^2}} = \frac{3}{2} H_0 dt. \quad (120)$$

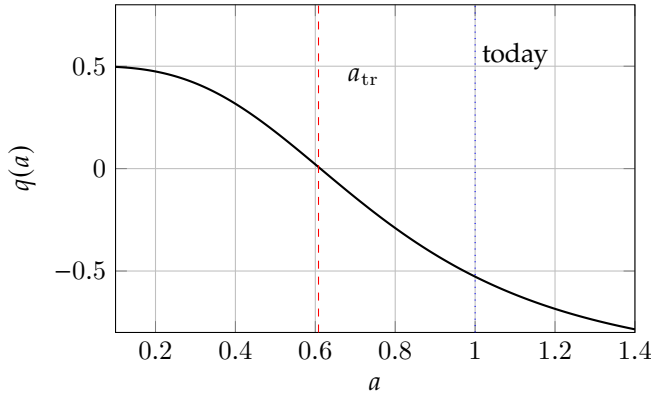


Figure 4. Deceleration parameter for flat matter plus Λ . The transition occurs when $q = 0$, corresponding to $z_{\text{tr}} \simeq 0.64$ for Planck-like parameters.

The integral is

$$\frac{1}{\sqrt{\Omega_\Lambda}} \sinh^{-1} \left(y \sqrt{\frac{\Omega_\Lambda}{\Omega_m}} \right) = \frac{3}{2} H_0 t. \quad (121)$$

Solving for $a = y^{2/3}$ gives

$$a(t) = \left(\frac{\Omega_m}{\Omega_\Lambda} \right)^{1/3} \sinh^{2/3} \left[\frac{3}{2} H_0 \sqrt{\Omega_\Lambda} t \right]. \quad (122)$$

This solution behaves as $t^{2/3}$ at early times and as $\exp(H_0 \sqrt{\Omega_\Lambda} t)$ at late times.

9. Age, Deceleration, and Distances

The age of the Universe in a model specified by $E(z)$ is

$$t_0 = \int_0^{t_0} dt = \frac{1}{H_0} \int_0^\infty \frac{dz}{(1+z)E(z)}. \quad (123)$$

Equivalently,

$$t(a) = \frac{1}{H_0} \int_0^a \frac{da'}{a' E(a')}. \quad (124)$$

For flat matter plus Λ , (122) gives

$$t_0 = \frac{2}{3H_0 \sqrt{\Omega_\Lambda}} \sinh^{-1} \sqrt{\frac{\Omega_\Lambda}{\Omega_m}}. \quad (125)$$

The lookback time to redshift z is

$$t_L(z) = t_0 - t(z) = \frac{1}{H_0} \int_0^z \frac{dz'}{(1+z')E(z')}. \quad (126)$$

Since $1+z = a^{-1}$, differentiation gives

$$\frac{dt}{dz} = -\frac{1}{(1+z)H(z)}. \quad (127)$$

This relation is used in chronometer measurements of $H(z)$, where differential ages of passively evolving galaxies provide an observational estimate of dz/dt .

The deceleration parameter is

$$q(t) \equiv -\frac{a\ddot{a}}{\dot{a}^2} = -\frac{\ddot{a}/a}{H^2}. \quad (128)$$

Using (58),

$$q = \frac{4\pi G}{3H^2} \left(\rho + \frac{3p}{c^2} \right) - \Omega_\Lambda. \quad (129)$$

For matter, radiation, and Λ ,

$$q = \frac{1}{2} \Omega_m + \Omega_r - \Omega_\Lambda. \quad (130)$$

At late times radiation is negligible, so

$$q_0 \simeq \frac{\Omega_{m0}}{2} - \Omega_{\Lambda 0}. \quad (131)$$

Acceleration begins when $q = 0$. For flat matter plus Λ ,

$$\frac{1}{2} \Omega_m(a_{\text{tr}}) = \Omega_\Lambda(a_{\text{tr}}). \quad (132)$$

This gives

$$a_{\text{tr}} = \left(\frac{\Omega_m}{2\Omega_\Lambda} \right)^{1/3}, \quad (133)$$

or

$$z_{\text{tr}} = \left(\frac{2\Omega_\Lambda}{\Omega_m} \right)^{1/3} - 1. \quad (134)$$

For light rays, $ds^2 = 0$. Radial propagation obeys

$$c \frac{dt}{a(t)} = d\chi. \quad (135)$$

The line-of-sight comoving distance is

$$D_C(z) = \frac{c}{H_0} \int_0^z \frac{dz'}{E(z')}. \quad (136)$$

The transverse comoving distance is

$$D_M = \begin{cases} \frac{c}{H_0 \sqrt{\Omega_k}} \sinh(\sqrt{\Omega_k} H_0 D_C / c), & \Omega_k > 0, \\ D_C, & \Omega_k = 0, \\ \frac{c}{H_0 \sqrt{|\Omega_k|}} \sin(\sqrt{|\Omega_k|} H_0 D_C / c), & \Omega_k < 0. \end{cases} \quad (137)$$

The angular-diameter distance is

$$D_A(z) = \frac{D_M(z)}{1+z}. \quad (138)$$

The luminosity distance is

$$D_L(z) = (1+z)D_M(z). \quad (139)$$

Equations (138) and (139) imply Etherington's reciprocity relation,

$$D_L = (1+z)^2 D_A, \quad (140)$$

valid whenever photons travel on null geodesics and photon number is conserved. The distance modulus is

$$\mu = m - M = 5 \log_{10} \left(\frac{D_L}{10 \text{ pc}} \right). \quad (141)$$

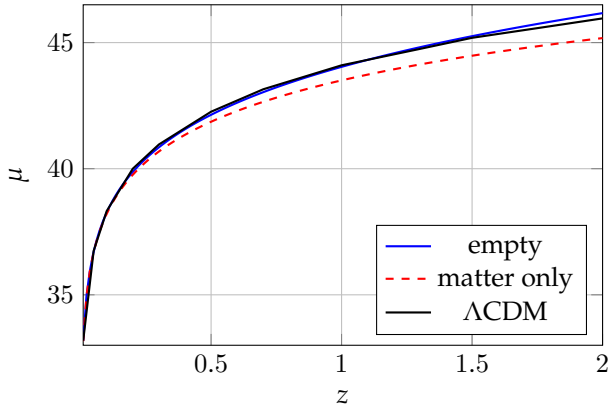


Figure 5. Qualitative Hubble diagram computed from luminosity-distance formulae. At fixed redshift, a flat accelerating Λ CDM universe predicts larger distances than an Einstein–de Sitter matter-only model, the central signature measured by Type Ia supernovae.

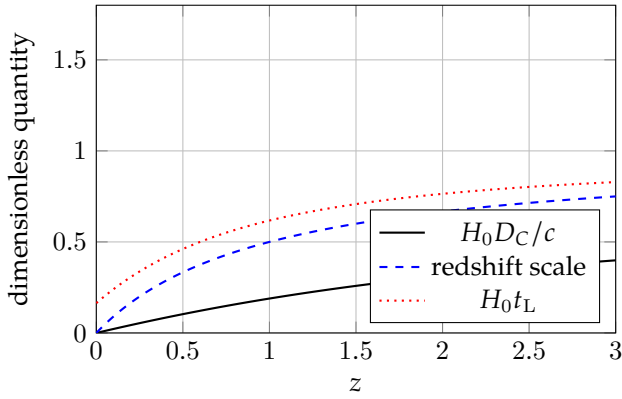


Figure 6. Dimensionless distance and lookback-time behavior for a flat Planck-like Λ CDM cosmology. Distances grow almost linearly at small z , while lookback time approaches a finite fraction of the Hubble time at high redshift.

Type Ia supernovae determine $\mu(z)$ and therefore directly probe the integral of the expansion history.

For a survey covering solid angle $d\Omega$, the comoving volume element is

$$\frac{dV_C}{dz d\Omega} = \frac{cD_M^2(z)}{H_0 E(z)}. \quad (142)$$

This quantity enters number-count tests using galaxies, clusters, and quasars.

10. Worked Numerical Applications

We adopt representative Planck 2018 baseline parameters [16]

$$H_0 = 67.4 \text{ km s}^{-1} \text{ Mpc}^{-1}, \quad \Omega_{m0} = 0.315, \quad (143)$$

and a nearly flat universe, so

$$\Omega_{\Lambda 0} \simeq 1 - \Omega_{m0} = 0.685. \quad (144)$$

The Hubble constant in SI units is

$$H_0 = 67.4 \frac{10^3 \text{ m s}^{-1}}{3.0856776 \times 10^{22} \text{ m}} \quad (145)$$

$$= 2.184 \times 10^{-18} \text{ s}^{-1}. \quad (146)$$

The Hubble time is

$$H_0^{-1} = 4.579 \times 10^{17} \text{ s} = 14.51 \text{ Gyr}. \quad (147)$$

The present critical density is

$$\rho_{c0} = \frac{3H_0^2}{8\pi G} \quad (148)$$

$$= \frac{3(2.184 \times 10^{-18} \text{ s}^{-1})^2}{8\pi(6.6743 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2})} \quad (149)$$

$$= 8.53 \times 10^{-27} \text{ kg m}^{-3}. \quad (150)$$

The matter density is

$$\rho_{m0} = \Omega_{m0} \rho_{c0} = 2.69 \times 10^{-27} \text{ kg m}^{-3}. \quad (151)$$

The analytic flat Λ CDM age is

$$t_0 = \frac{2}{3H_0 \sqrt{0.685}} \sinh^{-1} \sqrt{\frac{0.685}{0.315}} \quad (152)$$

$$= 0.951 H_0^{-1} = 13.80 \text{ Gyr}. \quad (153)$$

The present deceleration parameter is

$$q_0 = \frac{0.315}{2} - 0.685 = -0.528. \quad (154)$$

The transition redshift is

$$z_{\text{tr}} = \left(\frac{2(0.685)}{0.315} \right)^{1/3} - 1 \quad (155)$$

$$= 0.63. \quad (156)$$

This is the commonly quoted range $z_{\text{tr}} \simeq 0.6$ – 0.7 , with small shifts depending on radiation, curvature, and the precise dark-energy model.

The vacuum density corresponding to (144) is

$$\rho_{\Lambda 0} = 0.685 \rho_{c0} = 5.84 \times 10^{-27} \text{ kg m}^{-3}. \quad (157)$$

The associated cosmological constant is

$$\Lambda = \frac{8\pi G \rho_{\Lambda 0}}{c^2} \quad (158)$$

$$= 1.09 \times 10^{-52} \text{ m}^{-2}. \quad (159)$$

Matter–vacuum equality occurred at

$$z_{m\Lambda} = \left(\frac{0.685}{0.315} \right)^{1/3} - 1 \quad (160)$$

$$= 0.30. \quad (161)$$

The fact that acceleration began earlier, at $z_{\text{tr}} = 0.63$, reflects the factor of two in the condition $\rho_m = 2\rho_\Lambda$.

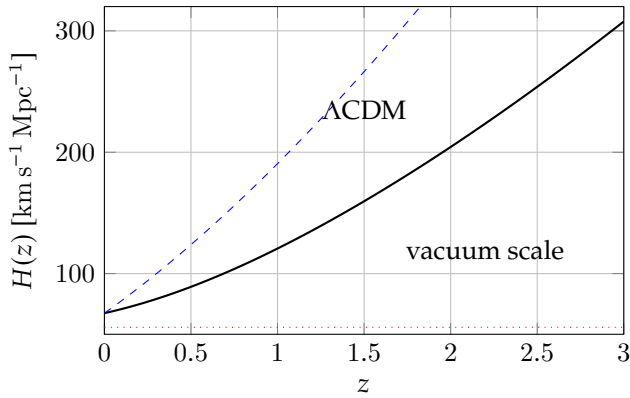
11. A Worked Distance Example: A Supernova at $z = 1$

Consider a Type Ia supernova at redshift $z = 1$ in the same flat Planck-like cosmology. Neglecting radiation at this redshift,

$$E(z) = \sqrt{0.315(1+z)^3 + 0.685}. \quad (162)$$

Table 1. Representative numerical quantities for a Planck 2018 flat Λ CDM cosmology.

Quantity	Value	Meaning
H_0	$67.4 \text{ km s}^{-1} \text{ Mpc}^{-1}$	expansion rate
Ω_{m0}	0.315	matter fraction
$\Omega_{\Lambda 0}$	0.685	vacuum fraction
H_0^{-1}	14.51 Gyr	Hubble time
ρ_{c0}	$8.53 \times 10^{-27} \text{ kg m}^{-3}$	critical density
t_0	13.80 Gyr	cosmic age
q_0	-0.528	present acceleration
z_{tr}	0.63	acceleration transition

**Figure 7.** Expansion rate for flat Λ CDM compared with the matter-only scaling. At high redshift matter dominates and $H \propto (1+z)^{3/2}$; at late times H approaches the vacuum scale $H_0\sqrt{\Omega_\Lambda}$.

The comoving distance is

$$D_C(1) = \frac{c}{H_0} \int_0^1 \frac{dz}{\sqrt{0.315(1+z)^3 + 0.685}}. \quad (163)$$

The Hubble distance is

$$\frac{c}{H_0} = \frac{2.99792458 \times 10^5 \text{ km s}^{-1}}{67.4 \text{ km s}^{-1} \text{ Mpc}^{-1}} \quad (164)$$

$$= 4.448 \times 10^3 \text{ Mpc}. \quad (165)$$

A Simpson-rule estimate with step $\Delta z = 0.25$ uses

$$I = \int_0^1 \frac{dz}{E(z)} \simeq \frac{\Delta z}{3} [f_0 + f_4 + 4(f_1 + f_3) + 2f_2], \quad (166)$$

where

$$f_i = \frac{1}{E(i\Delta z)}. \quad (167)$$

The five values are approximately

$$(f_0, f_1, f_2, f_3, f_4) = (1.000, 0.877, 0.756, 0.646, 0.559). \quad (168)$$

Thus

$$I \simeq \frac{0.25}{3} [1.000 + 0.559 + 4(0.877 + 0.646) + 2(0.756)] \quad (169)$$

$$= 0.765. \quad (170)$$

Therefore

$$D_C(1) = 0.765(4.448 \times 10^3 \text{ Mpc}) = 3.40 \times 10^3 \text{ Mpc}. \quad (171)$$

Flatness gives $D_M = D_C$, so the luminosity distance is

$$D_L(1) = (1+1)D_C(1) = 6.80 \times 10^3 \text{ Mpc}. \quad (172)$$

In parsecs,

$$D_L(1) = 6.80 \times 10^9 \text{ pc}. \quad (173)$$

The distance modulus follows from (141):

$$\mu(1) = 5 \log_{10} \left(\frac{6.80 \times 10^9 \text{ pc}}{10 \text{ pc}} \right) \quad (174)$$

$$= 44.16. \quad (175)$$

For a standardized Type Ia absolute magnitude $M \simeq -19.3$, the apparent magnitude would be

$$m = M + \mu \simeq 24.9. \quad (176)$$

This simple computation illustrates why high-redshift supernova cosmology requires large telescopes and careful calibration: the observational signal is encoded in magnitude differences of only a few tenths relative to alternative cosmologies.

The same redshift has a lookback time

$$t_L(1) = H_0^{-1} \int_0^1 \frac{dz}{(1+z)E(z)}. \quad (177)$$

Using the same Simpson nodes gives

$$H_0 t_L(1) \simeq 0.552, \quad (178)$$

or

$$t_L(1) \simeq 8.01 \text{ Gyr}. \quad (179)$$

The cosmic age at emission was consequently

$$t(z=1) \simeq 13.80 \text{ Gyr} - 8.01 \text{ Gyr} = 5.79 \text{ Gyr}. \quad (180)$$

12. Extensions and Observational Tests

The Friedmann equations are background equations. Their empirical power comes from the fact that many independent observations measure different integrals or derivatives of the same expansion history. Type Ia supernovae measure relative luminosity distances and are especially sensitive to the recent acceleration. Baryon acoustic oscillations measure combinations of $D_M(z)$ and $H(z)$ using the sound horizon as a standard ruler. Cosmic chronometers aim to measure $H(z)$ more directly through (127). The cosmic microwave background fixes the angular scale of the acoustic peaks and tightly constrains the physical matter and baryon densities [13, 16].

A useful CMB distance quantity is the angular acoustic scale,

$$\ell_A = \pi \frac{D_M(z_*)}{r_s(z_*)}, \quad (181)$$

where z_* is the redshift of photon decoupling and r_s is the comoving sound horizon. The sound horizon is

$$r_s(z_*) = \int_{z_*}^{\infty} \frac{c_s(z)}{H(z)} dz, \quad (182)$$

with photon-baryon sound speed

$$c_s(z) = \frac{c}{\sqrt{3[1 + R_b(z)]}}, \quad (183)$$

and

$$R_b(z) = \frac{3\rho_b}{4\rho_\gamma}. \quad (184)$$

Although detailed recombination physics is beyond the scope of this paper, these equations show why early-Universe physics and late-time geometry are linked in CMB parameter inference.

One simple extension replaces Λ by a dark-energy component with constant $w \neq -1$. In a flat universe with matter and constant- w dark energy,

$$E^2(z) = \Omega_{m0}(1+z)^3 + (1 - \Omega_{m0})(1+z)^{3(1+w)}. \quad (185)$$

The acceleration condition becomes

$$q(z) = \frac{1}{2}\Omega_m(z) + \frac{1+3w}{2}\Omega_{de}(z) < 0. \quad (186)$$

If $w < -1/3$, dark energy can drive acceleration; if $w = -1$, (186) reduces to the Λ CDM expression.

Curvature is another important generalization. Because $\Omega_k \propto a^{-2}/H^2$, it behaves like an effective fluid with

$$w_k = -\frac{1}{3}, \quad (187)$$

but it is geometric rather than material. Curvature does not directly appear in the acceleration equation (58); instead it changes the relation between distance and redshift through (137). This is why geometrical probes are particularly important for bounding Ω_k .

More radical extensions modify gravity itself. At the homogeneous level many such theories can be written in the suggestive form

$$H^2 = \frac{8\pi G}{3}(\rho + \rho_{\text{eff}}) - \frac{kc^2}{a^2}, \quad (188)$$

where ρ_{eff} represents new gravitational degrees of freedom or corrections to the Einstein-Hilbert action. However, matching the background expansion is not enough: a viable theory must also reproduce the observed growth of structure, gravitational lensing, and local tests of gravity [8, 9].

13. Discussion

The derivation shows that the Friedmann equations are not additional assumptions beyond general relativity and the cosmological principle. Once the FLRW metric is specified, homogeneity and isotropy reduce Einstein's ten field equations to two independent scalar

equations. The first Friedmann equation is a constraint equation relating the instantaneous expansion rate to density and curvature. The second is a dynamical equation for acceleration, showing explicitly that pressure gravitates.

Several assumptions underlie the construction. First, the Universe is treated as exactly homogeneous and isotropic. Real structure formation requires perturbations about the FLRW background; density contrasts grow gravitationally and eventually become nonlinear. Nevertheless, the background solution remains the correct zeroth-order description for distance-redshift relations, cosmic ages, and the mean expansion history. Second, matter components are modeled as perfect fluids. This is an excellent approximation for cold dark matter, baryons after averaging over small scales, photons before and after recombination, and a cosmological constant. More complex sources, such as scalar fields or imperfect fluids, can be included by replacing $p = w\rho c^2$ with the appropriate stress-energy tensor.

The observational importance of the acceleration equation cannot be overstated. If the Universe contained only matter and radiation with nonnegative pressure, then $\rho + 3p/c^2 > 0$ and (58) would imply deceleration. The supernova results of Riess et al. [14] and Perlmutter et al. [15] showed that distant Type Ia supernovae are dimmer than expected in a decelerating matter-dominated universe, implying larger luminosity distances and therefore accelerated expansion. In the minimal interpretation, this acceleration is caused by Λ , equivalent to a uniform vacuum energy with $w = -1$.

The Planck 2018 cosmic microwave background analysis gives a precise early-Universe determination of the parameters in the six-parameter flat Λ CDM model, including $H_0 = 67.4 \text{ km s}^{-1} \text{ Mpc}^{-1}$ and $\Omega_m = 0.315$ [16]. However, local distance-ladder measurements prefer a higher value, near $73 \text{ km s}^{-1} \text{ Mpc}^{-1}$, depending on calibration and sample selection. This discrepancy, known as the Hubble tension, is one of the central open problems in contemporary cosmology. It may reflect unrecognized systematics, or it may indicate physics beyond minimal Λ CDM, such as early dark energy, extra relativistic species, modified recombination, or more exotic gravitational physics. At present no proposed extension is universally accepted.

The cosmological constant itself raises a profound theoretical problem. Quantum field theory suggests that vacuum fluctuations should contribute to the stress-energy tensor, but naive estimates of the vacuum density exceed the observed value by enormous factors. Conversely, the observed density (157) is of the same order as the matter density today, even though matter and vacuum scale differently. This ‘‘coincidence problem’’ motivates dynamical dark-energy models, although current observations remain consistent with $w = -1$.

Finally, curvature is now strongly constrained. In the sum rule (89), Ω_k is close to zero observationally. Spa-

tial flatness is also a major prediction of inflationary cosmology, which drives $|\Omega_k|$ toward zero by accelerated expansion in the early Universe. Thus late-time acceleration and early-time inflation are mathematically related phenomena: both arise from a component with sufficiently negative effective pressure.

14. Conclusion

Starting from the cosmological principle, the FLRW metric is the unique spacetime geometry for an exactly homogeneous and isotropic universe. Direct calculation of its Christoffel symbols, Ricci tensor, Ricci scalar, and Einstein tensor leads to the Friedmann equations when the source is a comoving perfect fluid. The first Friedmann equation,

$$H^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} + \frac{\Lambda c^2}{3}, \quad (189)$$

is the expansion-rate constraint. The second,

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda c^2}{3}, \quad (190)$$

is the acceleration law. Covariant conservation gives

$$\dot{\rho} + 3H\left(\rho + \frac{p}{c^2}\right) = 0, \quad (191)$$

from which $\rho \propto a^{-3(1+w)}$ follows for constant equation-of-state parameter w .

Matter, radiation, and vacuum energy therefore have distinct expansion histories: $a \propto t^{2/3}$, $a \propto t^{1/2}$, and $a \propto e^{Ht}$, respectively. In the observed late Universe, a flat matter-plus- Λ model has the analytic solution

$$a(t) = \left(\frac{\Omega_m}{\Omega_\Lambda}\right)^{1/3} \sinh^{2/3}\left[\frac{3}{2}H_0\sqrt{\Omega_\Lambda}t\right], \quad (192)$$

which accurately describes the transition from decelerating matter domination to accelerating vacuum domination. The same framework explains the Type Ia supernova Hubble diagram, the age of the Universe, and the observed transition redshift near $z \approx 0.6$ – 0.7 . The Friedmann equations are therefore both a compact mathematical consequence of general relativity and the indispensable quantitative language of modern cosmology.

A. Additional Algebra for the Ricci Tensor

This appendix records intermediate steps that are often omitted in compact derivations. The trace of the mixed connection is

$$\Gamma^\alpha_{0\alpha} = \Gamma^i_{0i} = 3H. \quad (193)$$

Because $\Gamma^\alpha_{00} = 0$, the 00 Ricci component reduces to

$$R_{00} = -\partial_0\Gamma^\alpha_{0\alpha} - \Gamma^\alpha_{0\beta}\Gamma^\beta_{0\alpha}. \quad (194)$$

Using $\Gamma^i_{0j} = H\delta^i_j$ gives

$$\Gamma^\alpha_{0\beta}\Gamma^\beta_{0\alpha} = H^2\delta^i_j\delta^j_i = 3H^2, \quad (195)$$

and hence

$$R_{00} = -3\dot{H} - 3H^2 = -3\frac{\ddot{a}}{a}. \quad (196)$$

For spatial components, separate the four-dimensional result into an intrinsic three-curvature part and terms containing time derivatives of a . The intrinsic contribution is

$${}^{(3)}R_{ij} = 2k\gamma_{ij}. \quad (197)$$

The terms involving Γ^0_{ij} and Γ^i_{0j} combine as

$$\partial_0\Gamma^0_{ij} = \frac{a\ddot{a} + \dot{a}^2}{c^2}\gamma_{ij}, \quad (198)$$

$$\Gamma^0_{ij}\Gamma^\beta_{0\beta} = \frac{3a\dot{a}H}{c^2}\gamma_{ij}, \quad (199)$$

$$-2\Gamma^0_{it}\Gamma^\ell_{0j} = -\frac{2a\dot{a}H}{c^2}\gamma_{ij}. \quad (200)$$

Adding these time-dependent pieces gives

$$\frac{a\ddot{a} + 2\dot{a}^2}{c^2}\gamma_{ij}. \quad (201)$$

Combining with (197) yields

$$R_{ij} = \left(\frac{a\ddot{a}}{c^2} + \frac{2\dot{a}^2}{c^2} + 2k\right)\gamma_{ij}, \quad (202)$$

as quoted in (32).

B. Covariant Conservation in Components

The mixed tensor components for the comoving perfect fluid are

$$T^\mu_\nu = \text{diag}(\rho c^2, -p, -p, -p). \quad (203)$$

The $\nu = 0$ conservation law may be written

$$\nabla_\mu T^\mu_0 = \partial_\mu T^\mu_0 + \Gamma^\mu_{\mu\lambda}T^\lambda_0 - \Gamma^\lambda_{\mu 0}T^\mu_\lambda. \quad (204)$$

Only diagonal components contribute. The first term is

$$\partial_\mu T^\mu_0 = \frac{d}{dt}(\rho c^2). \quad (205)$$

The second term is

$$\Gamma^\mu_{\mu\lambda}T^\lambda_0 = \Gamma^\mu_{\mu 0}\rho c^2 = 3H\rho c^2. \quad (206)$$

The third term is

$$-\Gamma^\lambda_{\mu 0}T^\mu_\lambda = -\Gamma^i_{i0}T^i_i = 3Hp. \quad (207)$$

Thus

$$\nabla_\mu T^\mu_0 = c^2\dot{\rho} + 3H\rho c^2 + 3Hp. \quad (208)$$

Setting (208) to zero and dividing by c^2 gives

$$\dot{\rho} + 3H\left(\rho + \frac{p}{c^2}\right) = 0, \quad (209)$$

which is the fluid equation (61).

References

- [1] B. W. Carroll and D. A. Ostlie, *An Introduction to Modern Astrophysics*, 2nd ed., Addison-Wesley, San Francisco (2007).
- [2] A. Friedmann, Über die Krümmung des Raumes, *Zeitschrift für Physik* **10**, 377–386 (1922).
- [3] G. Lemaître, Un univers homogène de masse constante et de rayon croissant, *Annales de la Société Scientifique de Bruxelles* **47**, 49–59 (1927).
- [4] H. P. Robertson, Kinematics and world-structure, *Astrophysical Journal* **82**, 284–301 (1935).
- [5] A. G. Walker, On Milne’s theory of world-structure, *Proceedings of the London Mathematical Society* **42**, 90–127 (1937).
- [6] E. Hubble, A relation between distance and radial velocity among extra-galactic nebulae, *Proceedings of the National Academy of Sciences* **15**, 168–173 (1929).
- [7] A. Einstein, Kosmologische Betrachtungen zur allgemeinen Relativitätstheorie, *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften* (1917), 142–152.
- [8] S. Weinberg, *Gravitation and Cosmology*, Wiley, New York (1972).
- [9] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation*, W. H. Freeman, San Francisco (1973).
- [10] P. J. E. Peebles, *Principles of Physical Cosmology*, Princeton University Press, Princeton (1993).
- [11] S. Dodelson, *Modern Cosmology*, Academic Press, San Diego (2003).
- [12] B. Ryden, *Introduction to Cosmology*, 2nd ed., Cambridge University Press, Cambridge (2017).
- [13] W. Hu and N. Sugiyama, Small-scale cosmological perturbations: an analytic approach, *Astrophysical Journal* **471**, 542–570 (1996).
- [14] A. G. Riess et al., Observational evidence from supernovae for an accelerating universe and a cosmological constant, *Astronomical Journal* **116**, 1009–1038 (1998).
- [15] S. Perlmutter et al., Measurements of Ω and Λ from 42 high-redshift supernovae, *Astrophysical Journal* **517**, 565–586 (1999).
- [16] Planck Collaboration, N. Aghanim et al., Planck 2018 results. VI. Cosmological parameters, *Astronomy & Astrophysics* **641**, A6 (2020).